Abstract Interpretation (1/2)

Martin Kellogg

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B. inlining
C. refinement
D. concretization

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When dealing with a concrete language, we don't usually get to **choose** the domain or the semantics. But in abstract interpretation, we do!

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  - an ***abstract domain*** is a layer of indirection on top of the concrete domain that splits the concrete domain into a smaller number of sets

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Important property of an abstract domain: it must **completely cover** the concrete domain

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  - each $\boldsymbol{A_i}$ represents an ***abstract value***
    - e.g., "odd integers", "Strings that match my regular expression", etc.

Domains: orderings and lattices

- An abstract domain is incomplete without an **ordering**: that is, a way to tell how the abstract values are related to each other
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- There are two ways to express the ordering:
  - define a **less than relation** (usually denoted by $\sqsubset$), or
  - define a **least upper bound operator** (usually denoted by $\sqcup$)
- These two approaches are **equivalent**: you can derive the LUB from the less than relation and vice-versa

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  - formally, we define a relation as a set of ordered pairs
- If $x \sqsubset y$, then we say that x is lower or less, and that y is higher or greater
- The less-than relation **need not be total**
  - for two points $e1$ and $e2$, it is possible that neither $e1 \sqsubset e2$ nor $e2 \sqsubset e1$ is true

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- The least upper bound is often more useful, because it directly models the **join operator**
  - that is, it models what happens when two possible abstract values flow to the same location (e.g., the then and else branches of an if)

Least upper bound: relationship to types

- You are probably already intuitively familiar with the LUB operator from your experience with **object-oriented programming**

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              Object
             /      \
        Animal        Shape
       /      |         |   \
   Bird    Mammal    Circle  Rect
            /  \               |
         Dog    Cat          Square
```

Least upper bound: relationship to types

- You are probably already intuitively familiar with the LUB operator from your experience with **object-oriented programming**
  - any time that you've answered the question "what is the closest supertype that these two types share", you're doing a LUB

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    - LUB is a binary function; for a binary function f, monotonicity is defined as
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    - Note that this is not the same as:
      - $\forall\ x, y\ .\ f(x, y) \sqsupseteq x \wedge f(x, y) \sqsupseteq y$!
      - though this property is also true of the LUB operator

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Hint: I like to ask exam questions like "why is this property required?" or "what would happen if it weren't true?"

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A set is ***partially ordered*** iff ∃ a binary relationship ≤ that is:

- ***reflexive***: x ≤ x
- ***anti-symmetric***: x ≤ y ∧ y ≤ x => x = y
- ***transitive***: x ≤ y ∧ y ≤ z => x ≤ z

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- We saw some examples of lattices last week
  - e.g., the null pointer analysis example's lattice with T, c, and ⊥

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- the goal of the transfer functions are to encode the ***abstract semantics*** of the operations in the programming language
  - that is, the transfer function for an operation answers the question "what does this operation ***mean*** in the context of the abstract domain"?
- formally, an abstract interpretation requires a transfer function for **each language construct**
  - in practice, though, we usually assume that most are obvious and focus on the ones that might be interesting, which is what I'll do in the examples on the next few slides

Example AI: even/odd integers

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A note about top:

- top represents ***no constraints*** on the possible values
- equivalently, ***every value*** is a member of top

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Similarly for bottom:

- bottom represents ***all possible constraints at once*** on values
- equivalently, ***no values*** are members of bottom

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Example transfer function:

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even	T	even	odd	⊥
odd	T	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

Example AI: even/odd integers

Let’s apply this AI to an example:

```
x = 0;
y = read_even();
x = y + 1;
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Concrete execution

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Abstract interpr.

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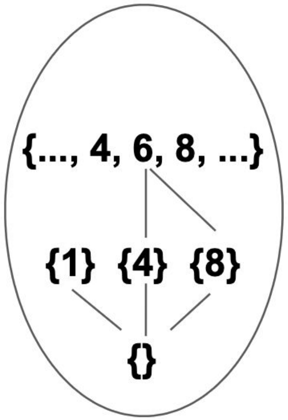

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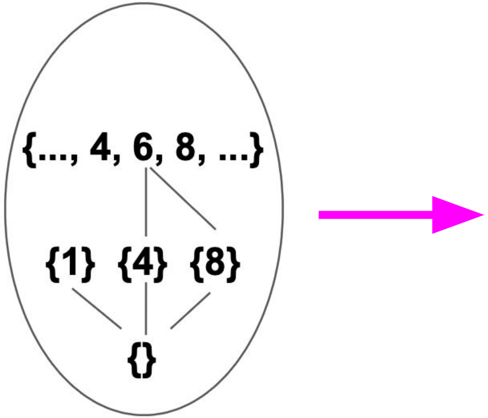

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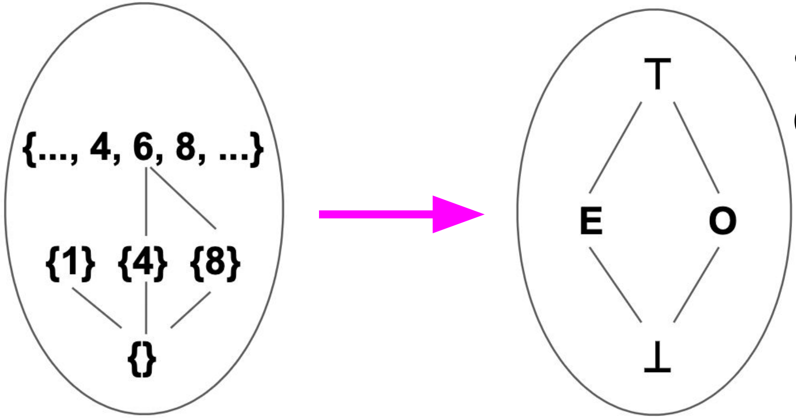

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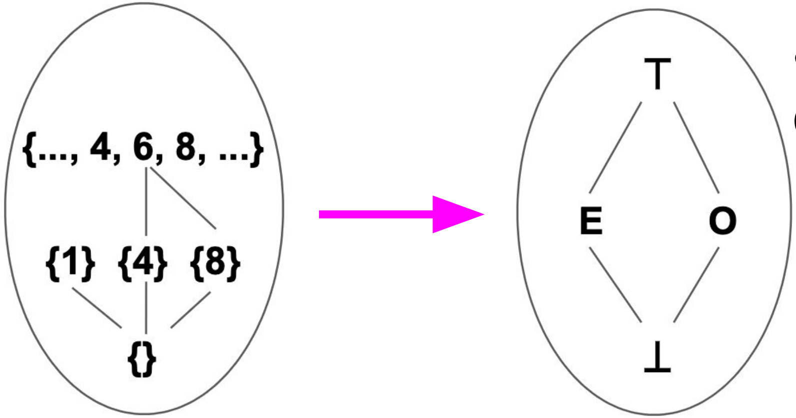

abstract domain

e.g.:
α(4) = even
α({}) = bottom

Concretization function

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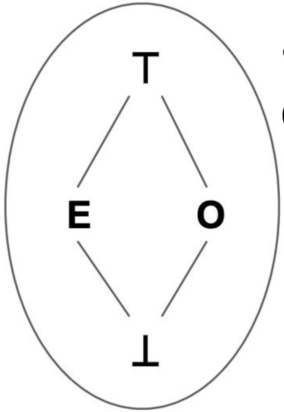

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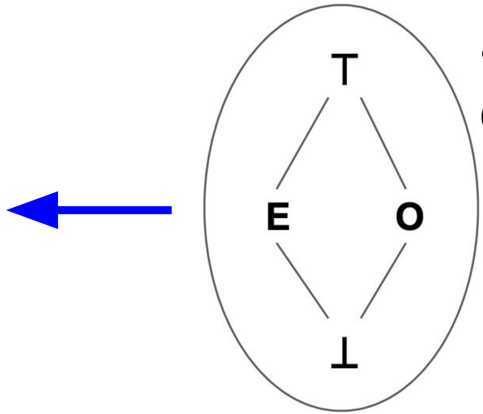

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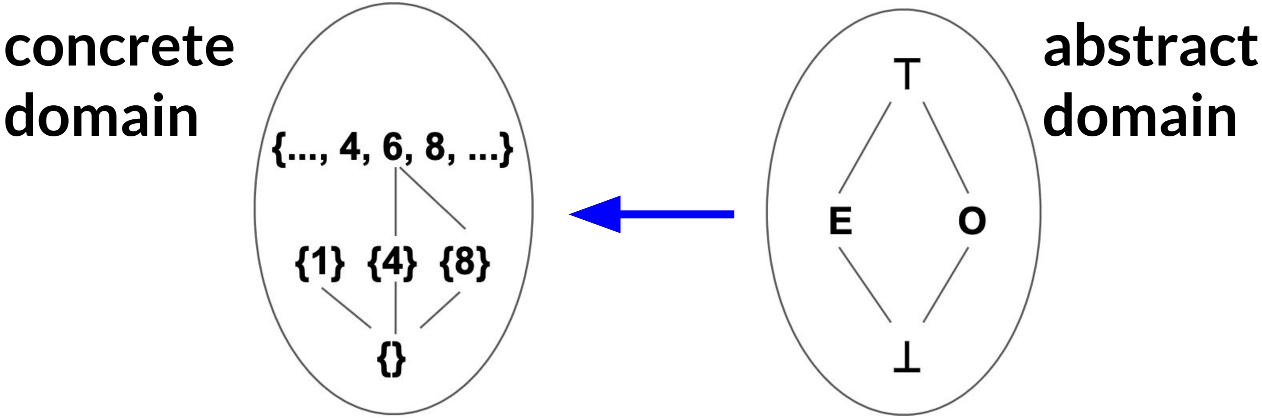

Role of abstr., concr., and transfer fcns.

Concrete state

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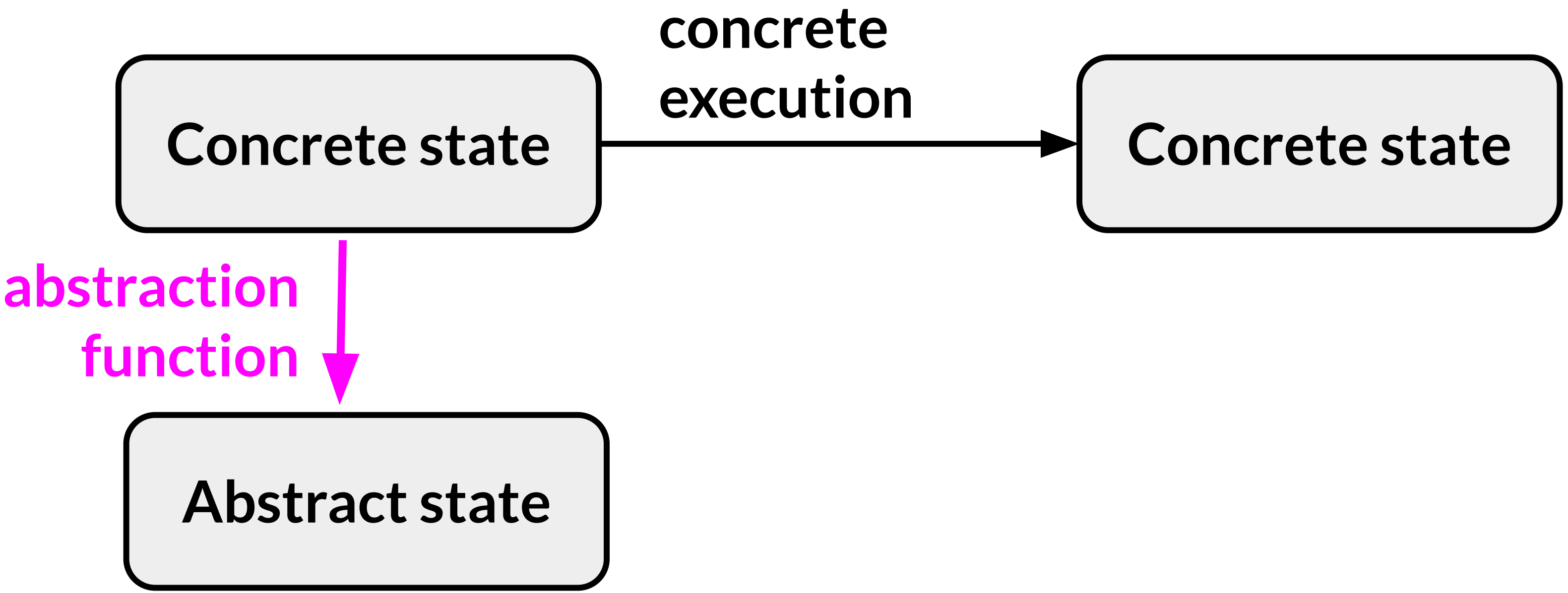

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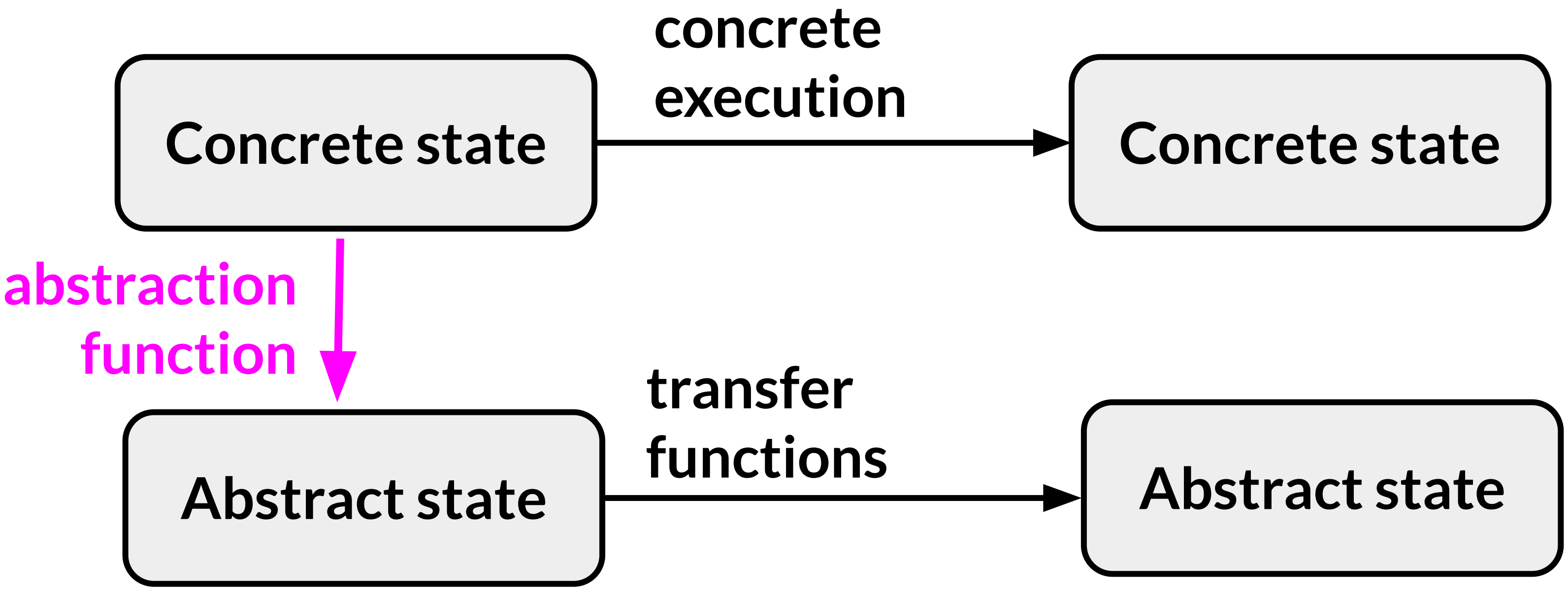

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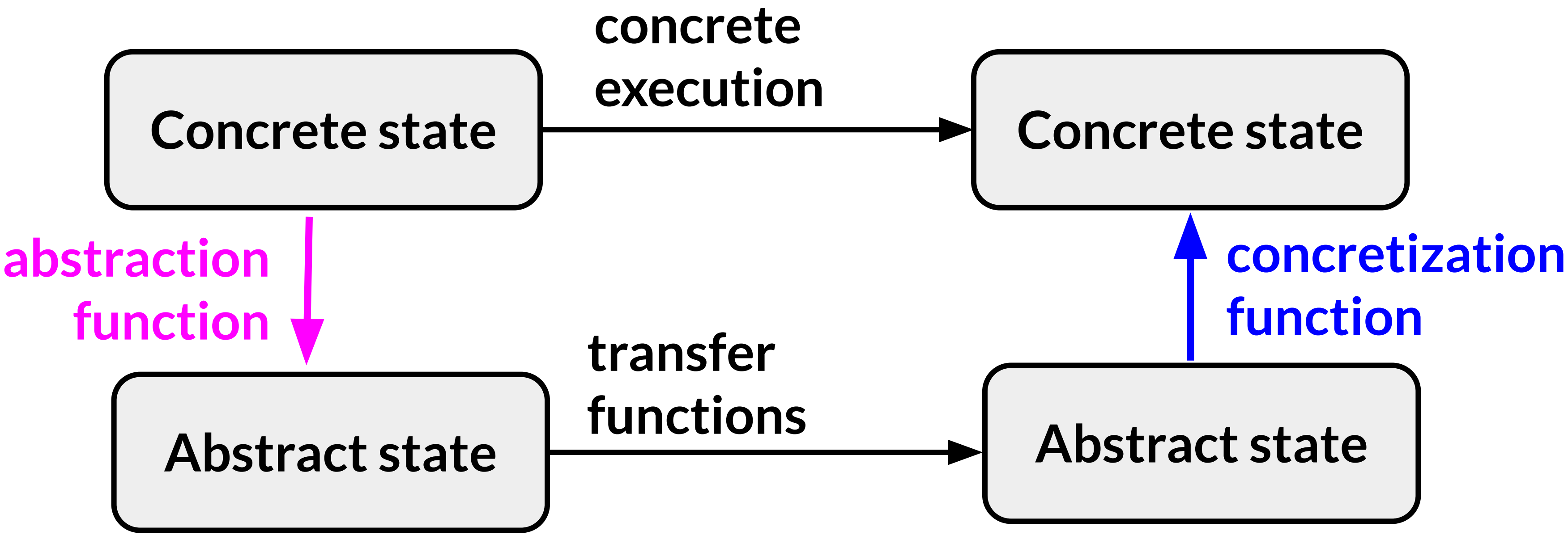

Example AI: even/odd integers

Let's apply this AI to an example:

```
x = 0;
y = read_even()
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Concrete execution		Abstract interpr.	
{x=0;	y=undef}	{x=**e**;	y=?}
{x=0;	y=8}	{x=?;	y=?}
{x=9;	y=8}	{x=?;	y=?}
{x=9;	y=18}	{x=?;	y=?}
{x=16;	y=18}	{x=?;	y=?}
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{x=0;	y=undef}	{x=**e**;	y=⊥}
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{x=9;	y=8}	{x=?;	y=?}
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Example AI: even/odd integers

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transfer function for +!

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Example AI: even/odd integers

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Example AI: even/odd integers

What's the transfer function for division?

↓/→	⊤	even	odd	⊥
⊤				
even				
odd				
⊥				

Example AI: even/odd integers

What's the transfer function for division?

↓/→	T	even	odd	⊥
T	T	T	T	⊥
even	T	T	T	⊥
odd	T	T	T	⊥
⊥	⊥	⊥	⊥	⊥

Notes for online readers:

- even/even is top:
  - 6/2 = 3
  - 8/2 = 4
- odd/odd is top:
  - 5/5 = 1
  - 11/5 = 2
    - integer division!

Example AI: even/odd integers

Let's apply this AI to an example:

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{x=9; y=18}	{x=**o**; y=**e**}
{x=16; y=18}	{x=**e**; y=**e**}
{x=16; y=8}	{x=**e**; y=**T**}

Example AI: even/odd integers

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for x, our abstraction was precise

Example AI: even/odd integers

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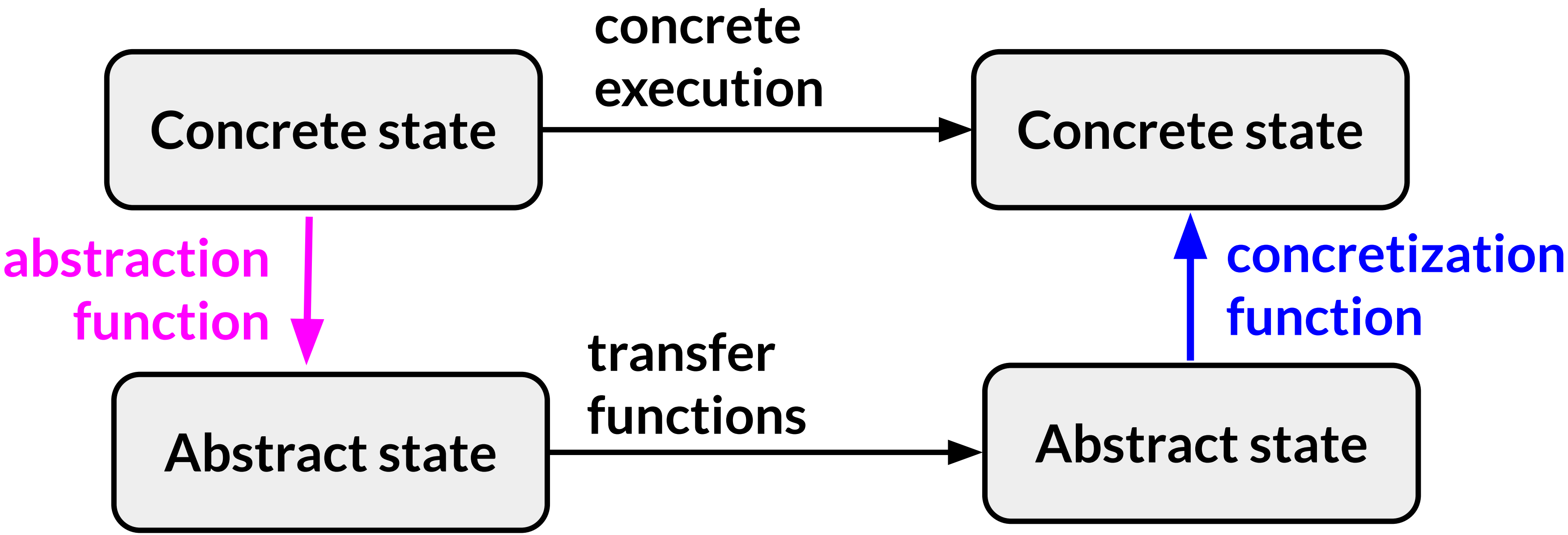
{x=9; y=18}	{x=**o**; y=**e**}
{x=16; y=18}	{x=**e**; y=**e**}
{x=16; y=8}	{x=**e**; y=**⊤**}

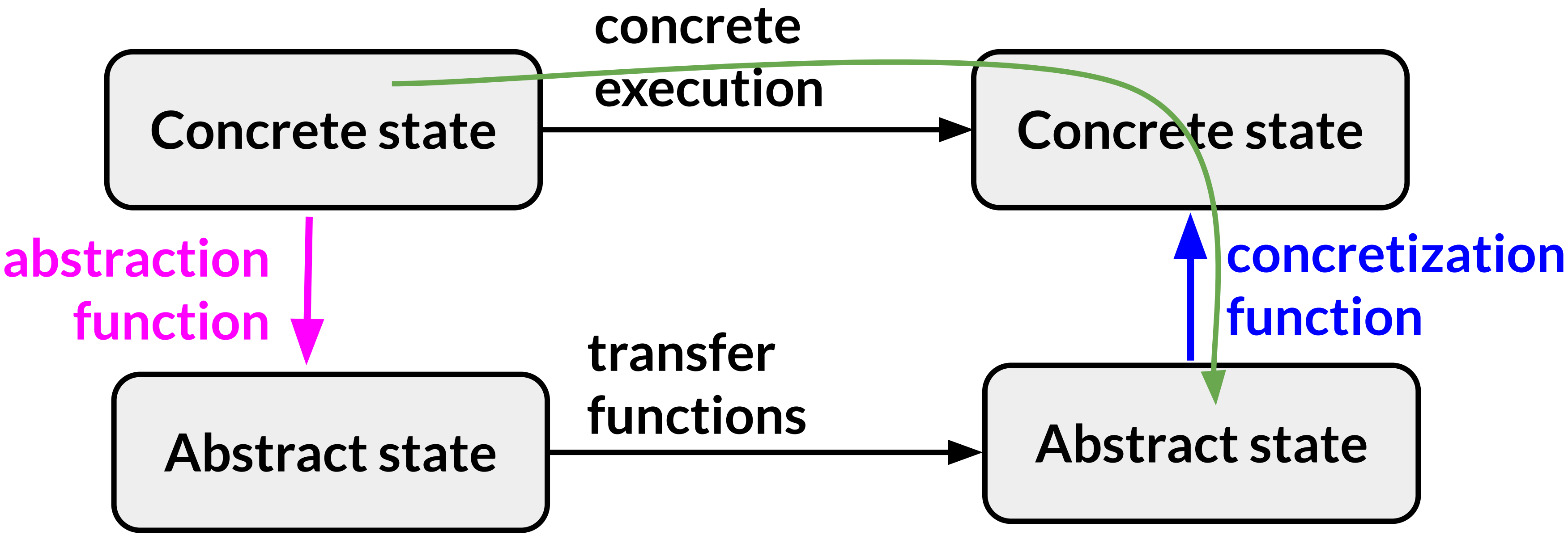
for x, our abstraction was precise

but for y, it was not

Approximation!



Approximation!



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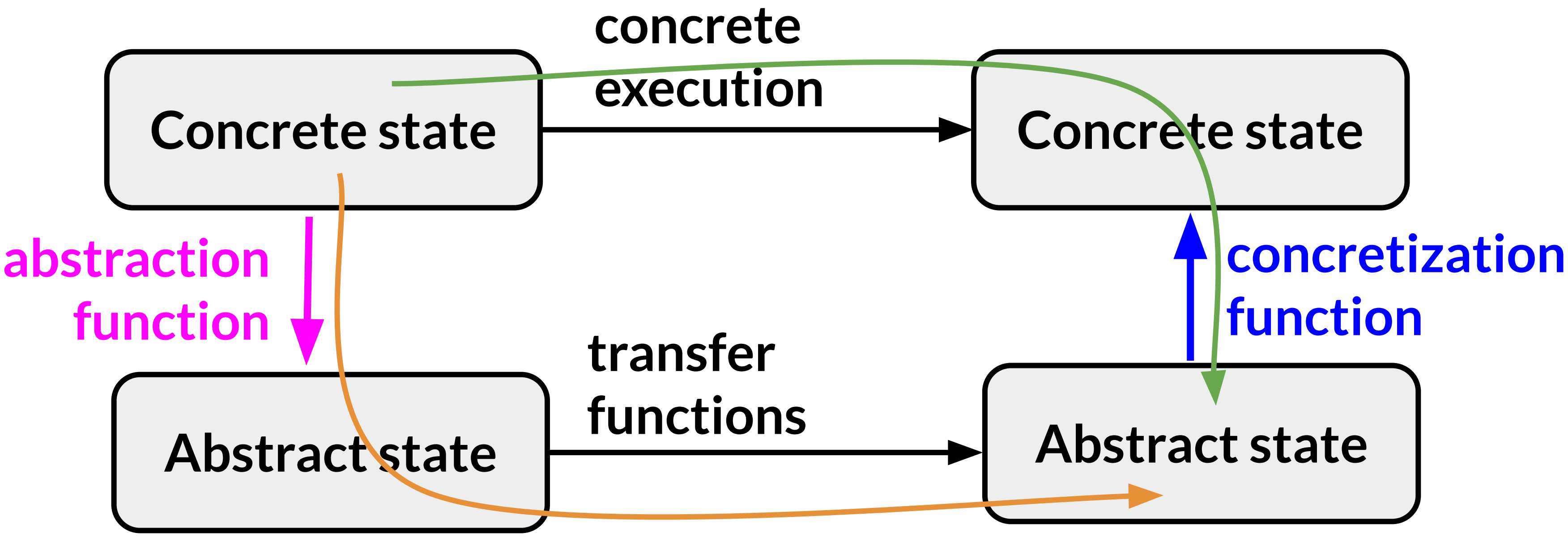

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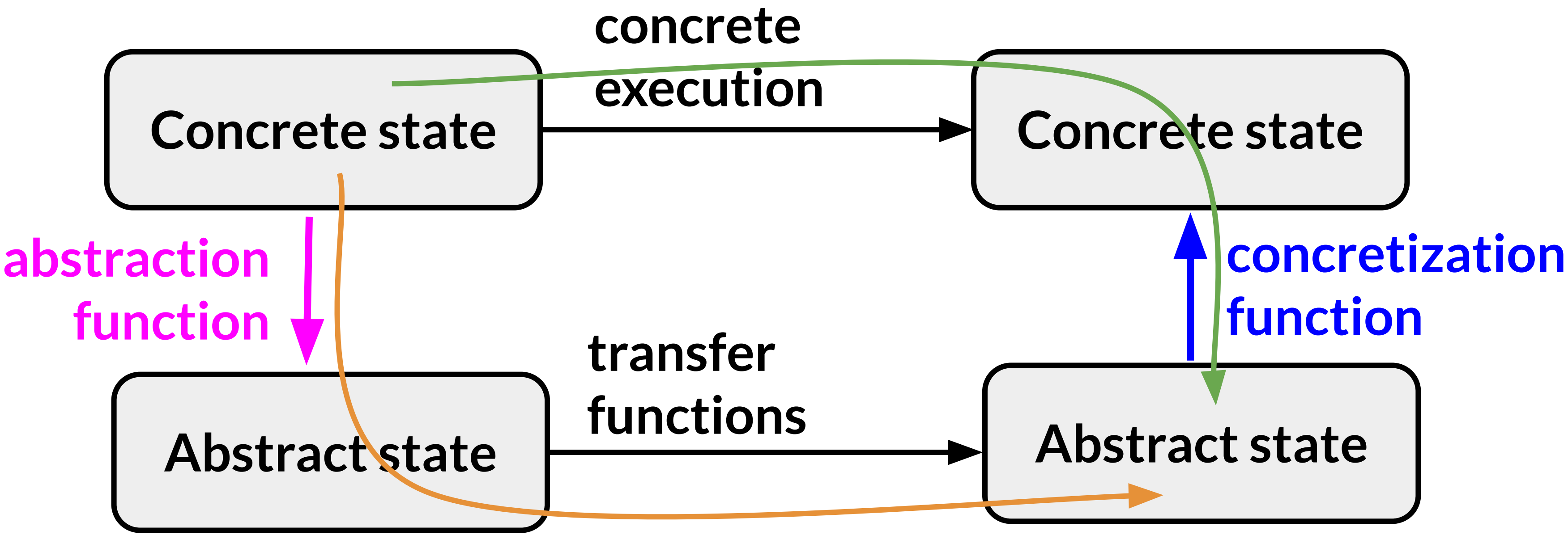
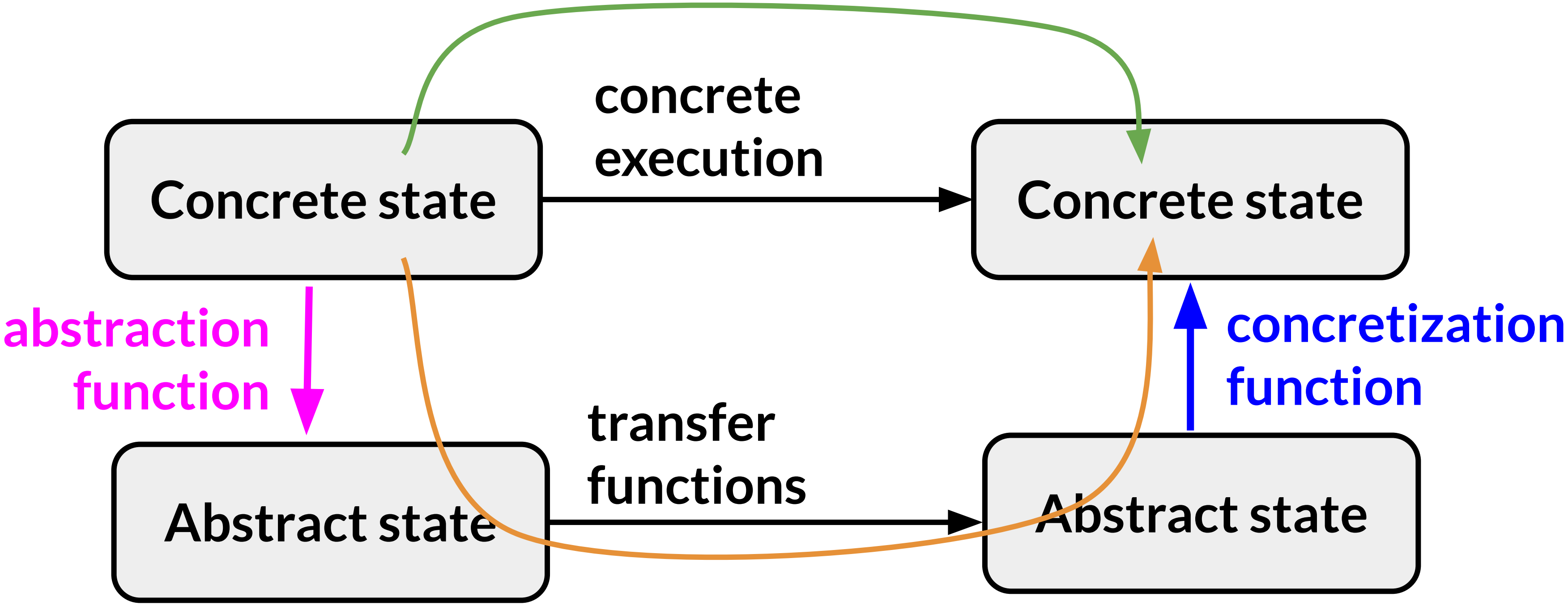


Do the green and orange paths always lead to the same abstract state?

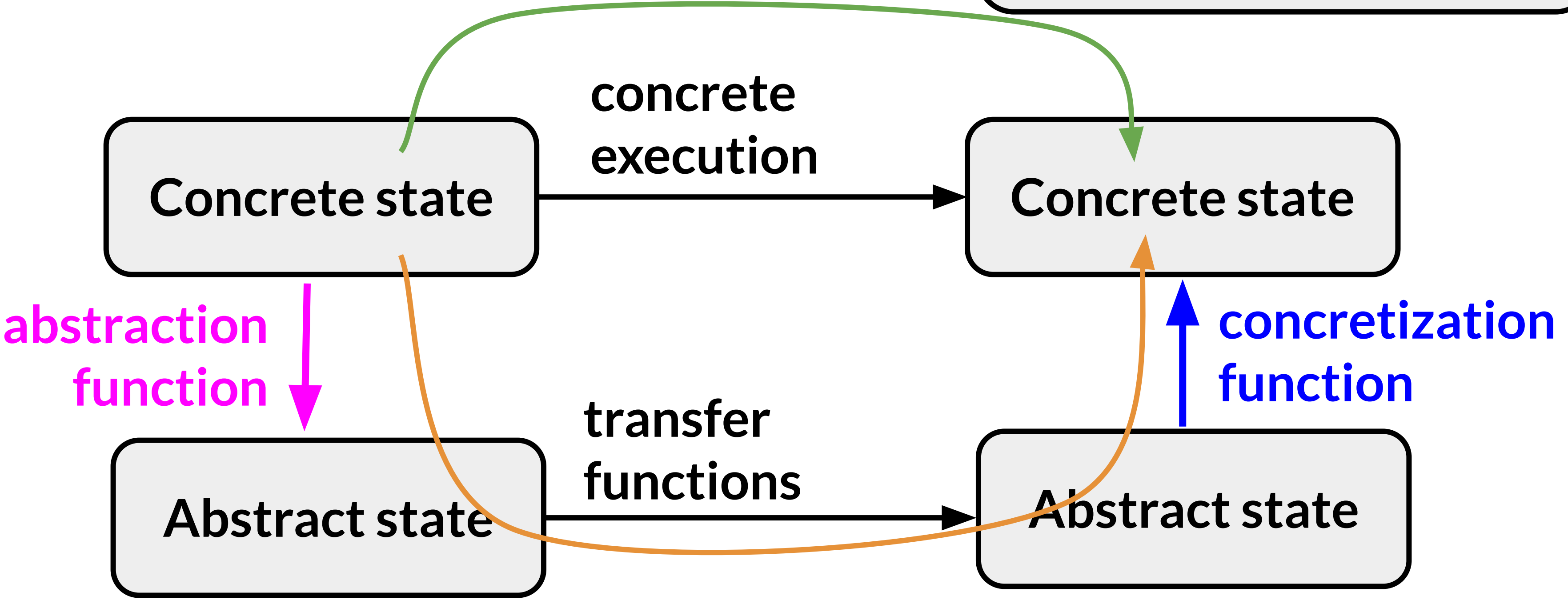
Approximation!



Do the green and orange paths always lead to the same concrete state?

Approximation!

We'll come back to this question when we discuss **soundness**



Do the green and orange paths always lead to the same concrete state?

Alternative example AI: even/odd integers

Is there an **alternative** AI that we can use to conclude that y is even after we analyze the example?

```
x = 0;
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y = x / 2;
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Alternative example AI: even/odd integers

Is there an alternative AI that we can use to conclude that y is even after we analyze the example?

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

In-class exercise: with a partner, ***design an alternative*** abstract interpretation that can conclude that y is even.

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

Alternative example AI: even/odd integers

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- ask yourself: "**for what x is that true?**"

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Alternative example AI: even/odd integers

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  - alternative answer: abstract value tracks the number of 2s in the prime factorization

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

- ask yourself: "**for what x is that true?**"
  - simplest answer: `x . x%4 = 0` - that is, all `x`s such that x is **divisible by 4**
  - alternative answer: abstract value tracks the number of 2s in the prime factorization
- cunning plan: add a "divisible by 4" abstract value (**mod4**) to our lattice, then rebuild our transfer functions

Alternative example AI: even/odd integers

Next question: where does “divisible by 4” go in the **lattice**?

```
{ even, odd } = top
     /           \
  {even}       {odd}
     \           /
      {} = bottom
```

Alternative example AI: even/odd integers

Next question: where does “divisible by 4” go in the **lattice**?

```
      { even, odd } = top
          /            \
     {even}        {odd}
         |              |
    {mod4}            |
         \             /
          {} = bottom
```

all mod4 integers are also even!

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

Alternative example AI: even/odd integers

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recall our **original** transfer function for +:

+	⊤	even	odd	⊥
⊤	⊤	⊤	⊤	⊥
even	⊤	even	odd	⊥
odd	⊤	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

recall our **original** transfer function for +:

we need to add a row and a column for **mod4**:

+	⊤	even	odd	mod4	⊥
⊤	⊤	⊤	⊤		⊥
even	⊤	even	odd		⊥
odd	⊤	odd	even		⊥
mod4					
⊥	⊥	⊥	⊥		⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

recall our **original** transfer function for +:

we need to add a row and a column for **mod4**:

+	T	even	odd	mod4	⊥
T	T	T	T	T	⊥
even	T	even	odd	even	⊥
odd	T	odd	even	odd	⊥
mod4	T	even	odd	mod4	⊥
⊥	⊥	⊥	⊥	⊥	⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

same thing for **division**:

$\downarrow/\rightarrow$	$\top$	even	odd	mod4	$\bot$
$\top$	$\top$	$\top$	$\top$		$\bot$
even	$\top$	$\top$	$\top$		$\bot$
odd	$\top$	$\top$	$\top$		$\bot$
mod4					
$\bot$	$\bot$	$\bot$	$\bot$		$\bot$

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

same thing for **division**:

oh no! why is mod4 divided by even top?

- 4/4 = 1 :(
- we need **another** lattice element to make this work!

↓/→	⊤	even	odd	mod4	⊥
⊤	⊤	⊤	⊤	⊤	⊥
even	⊤	⊤	⊤	⊤	⊥
odd	⊤	⊤	⊤	⊤	⊥
mod4	⊤	⊤	⊤	⊤	⊥
⊥	⊥	⊥	⊥	⊥	⊥

Alternative example AI: even/odd integers

Another lattice element: “**is2**”

Alternative example AI: even/odd integers

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- **sibling** of mod4 in the lattice

Alternative example AI: even/odd integers

Another lattice element: "**is2**"

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```
       { even, odd } = top
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      {even}       {odd}
      /     \         |
{mod4}  {is2}         |
    \      |         /
      {} = bottom
```

Alternative example AI: even/odd integers

Another lattice element: “is2”

- sibling of mod4 in the lattice
- its only purpose is to be treated specially in the division transfer function

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      { even, odd } = top
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     {even}       {odd}
      /      \          |
{mod4}  {is2}     |
     \        |       /
      {} = bottom
```

Alternative example AI: even/odd integers

Another lattice element: "**is2**"

- **sibling** of mod4 in the lattice
- its only purpose is to be treated specially in the **division transfer function**
  - in particular, we add the rule "**mod4 / is2 -> even**"
  - full transfer functions left as an exercise

```
        { even, odd } = top
           /           \
      {even}        {odd}
      /      \          |
{mod4} {is2}        |
      \         |       /
       {} = bottom
```

Alternative example AI: let's try it

```
x = 0;
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y = x / 2;
```

Abstract interpr.

```
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
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{x=e;    y=⊥}
{x=e;    y=e}
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what should the transfer function for even - is2 be?

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- **even! why not mod4?**

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what should the transfer function for even - is2 be?

- **even! why not mod4? counterexample: 8 - 2 = 6**

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{x=**o**;	y=**e**}
{x=**e**;	y=**e**}
{x=**e**;	y=**T**}

Alternative example AI: even/odd integers

- Why did adding **is2** and **mod4** fail to fix the approximation problem in the example?

Alternative example AI: even/odd integers

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  - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
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- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**! Keep this in mind on HW8.

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- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**! Keep this in mind on HW8.
- how could we get the right answer on this example?
  - more complex abstract values, e.g., oddTimes2?
  - store the mathematical expression for each variable?

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  - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
    - and if X is initially even, then this means that the result is divisible by 4
- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**! Keep this in mind on HW8.
- how could we get the right answer on this example?
  - more complex abstract values, e.g., oddTimes2?
  - store the mathematical expression for each variable?

one more try…

Alternative example AI: even/odd integers

Yet another lattice element: “**odd2**”

```
      { even, odd } = top
         /           \
   {even}        {odd}
    /       \          |
{mod4} {is2}       |
     \       |       /
      {} = bottom
```

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- **where does it go** in the lattice?
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    - between even and is2!

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        - odd2 - is2 -> mod4

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Alternative example AI: another attempt

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Abstract interpr.

```
{x=?;    y=?}
{x=?;    y=?}
{x=?;    y=?}
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{x=?;     y=?}
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Success!

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Using LUB at join points models the fact that the program may **take either branch** of an if statement.

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   c. if either a. or b. caused a change, re-add dependent blocks to the worklist

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You may be surprised that it is possible to build an abstract interpretation using (some) infinite-height lattices. Next week, we'll discuss ***widening***, which is the technique for this.

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- otherwise, loops are just a join point and a back-edge in the CFG

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- **pessimistic** algorithms are also possible
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  - can be stopped at any time (e.g., when a budget is reached), but answer may not be precise

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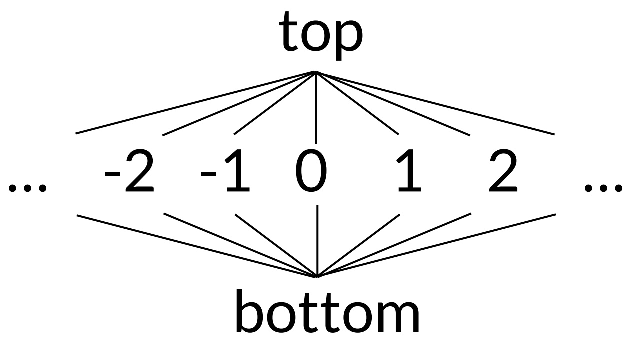

Another example

Consider the following program:

```
w = 5
x = read()
if (x is even)
  y = 5
  w = w + y
else
  y = 10
  w = y
z = y + 1
x = 2 * w
```

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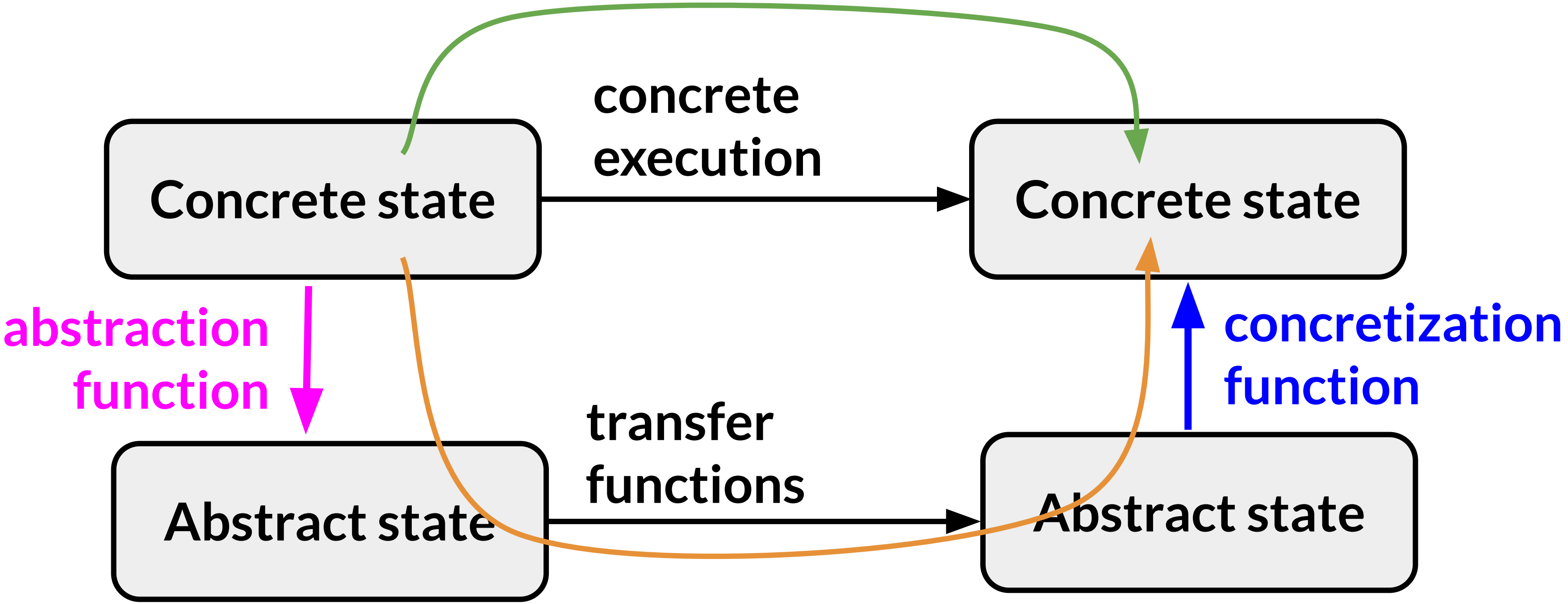
And, it's also necessary to show that the Galois connection holds for the **transfer functions**!

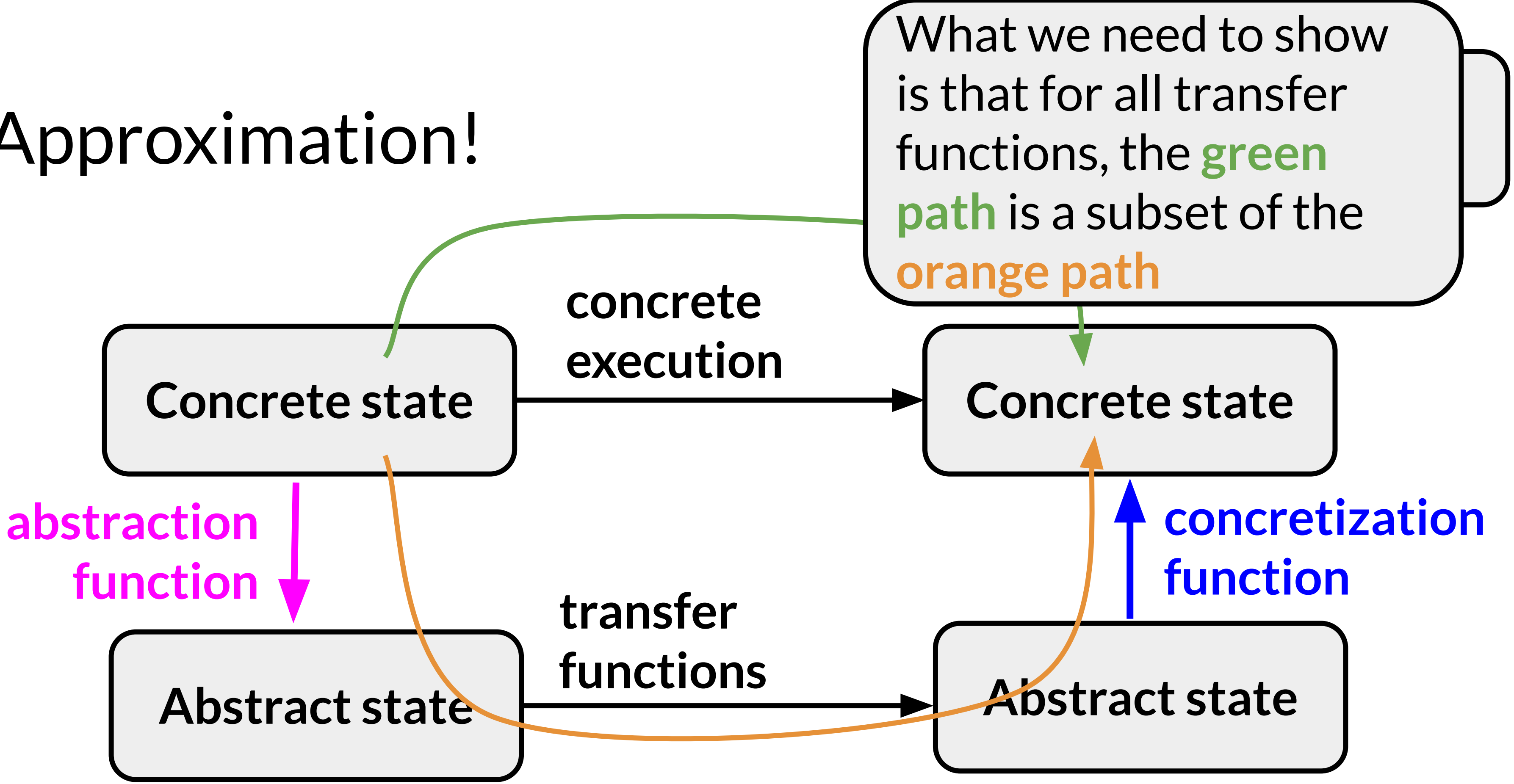
Approximation!

Remember this diagram from earlier?



Do the green and orange paths always lead to the same concrete state?

Approximation!



Do the green and orange paths always lead to the same concrete state?

Course announcements

- If you have not yet collected your exam, I have it at the front
- This week's homework is **individual** (you may not work with a partner)
    - this is a difference from previous homeworks!
- Next week's homework:
    - builds on this week's - if you don't do this week's homework, you will not be able to do next week's
    - is also individual
- This week's homework involves designing an abstract interpretation. Keep in mind the pitfalls that we talked about today!