Abstract Interpretation (2/2)

Martin Kellogg

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    - usually represented as tables

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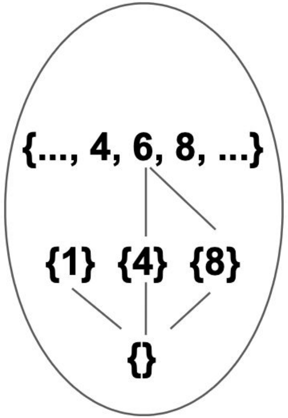

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- the *concrete domain* of a variable is the set of values that the variable might actually take on during execution
- an *abstract domain* is a layer of indirection on top of the concrete domain that splits it into a smaller number of sets

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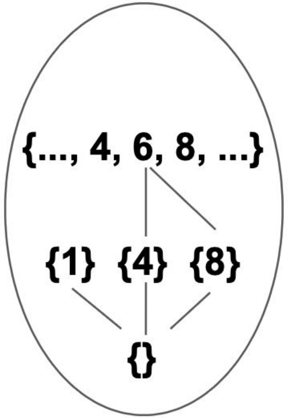

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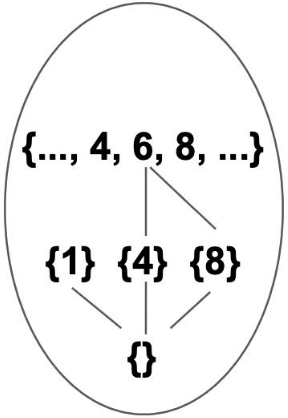

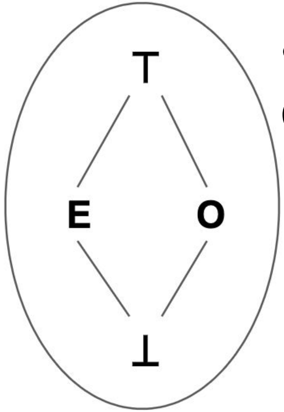

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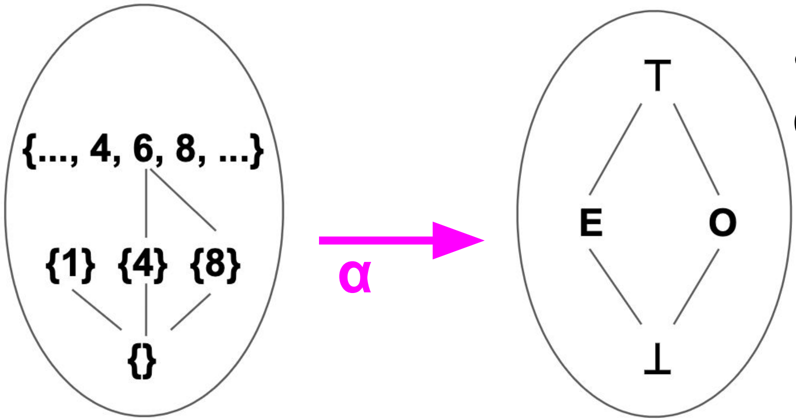

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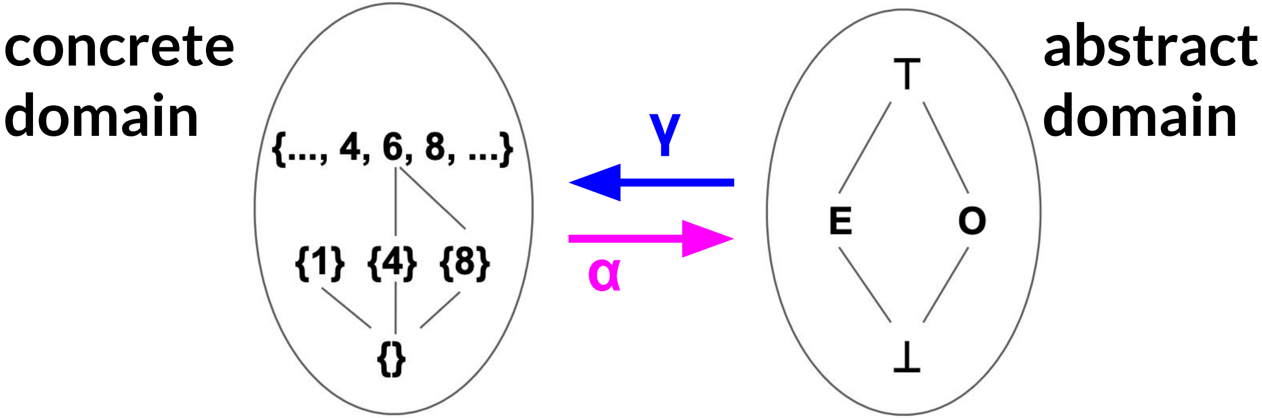

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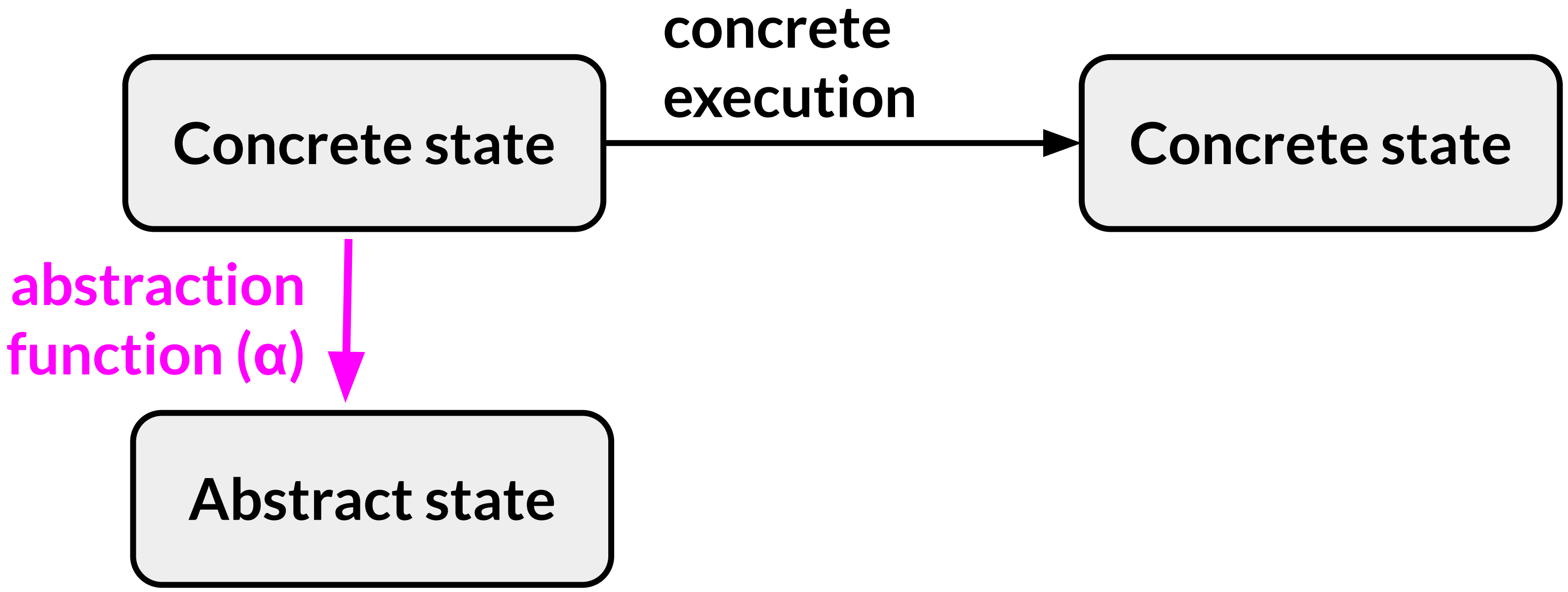

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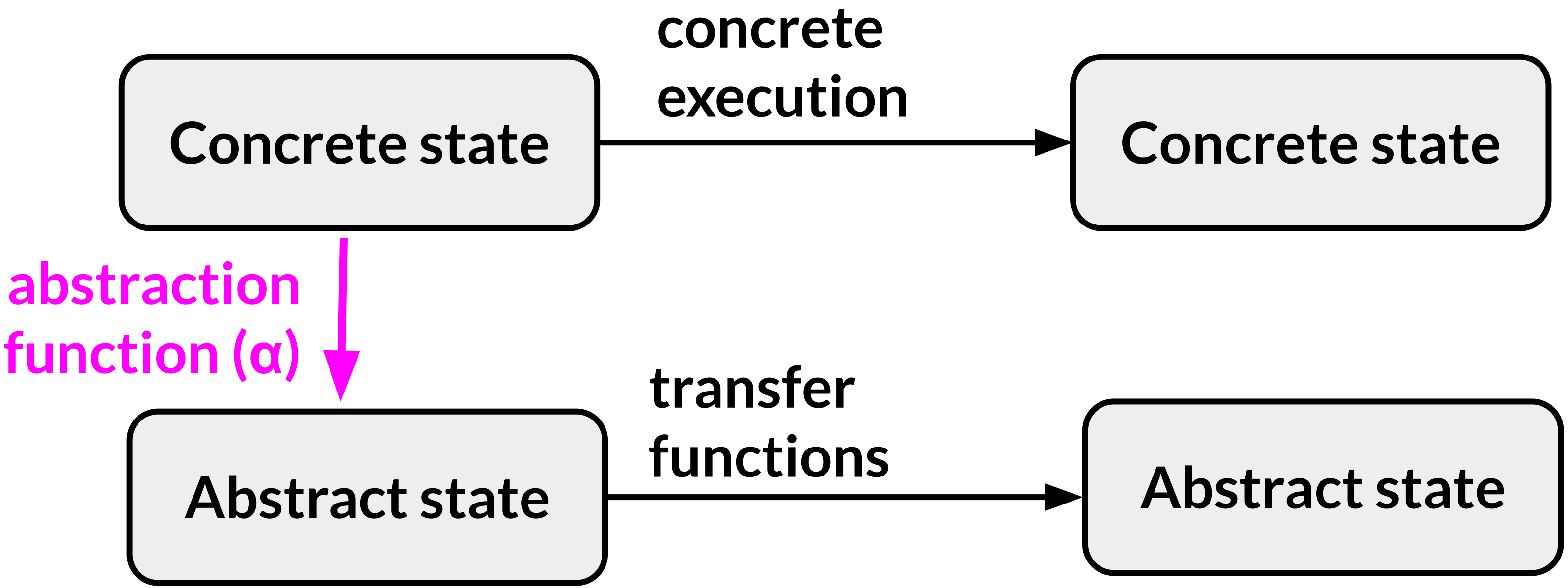

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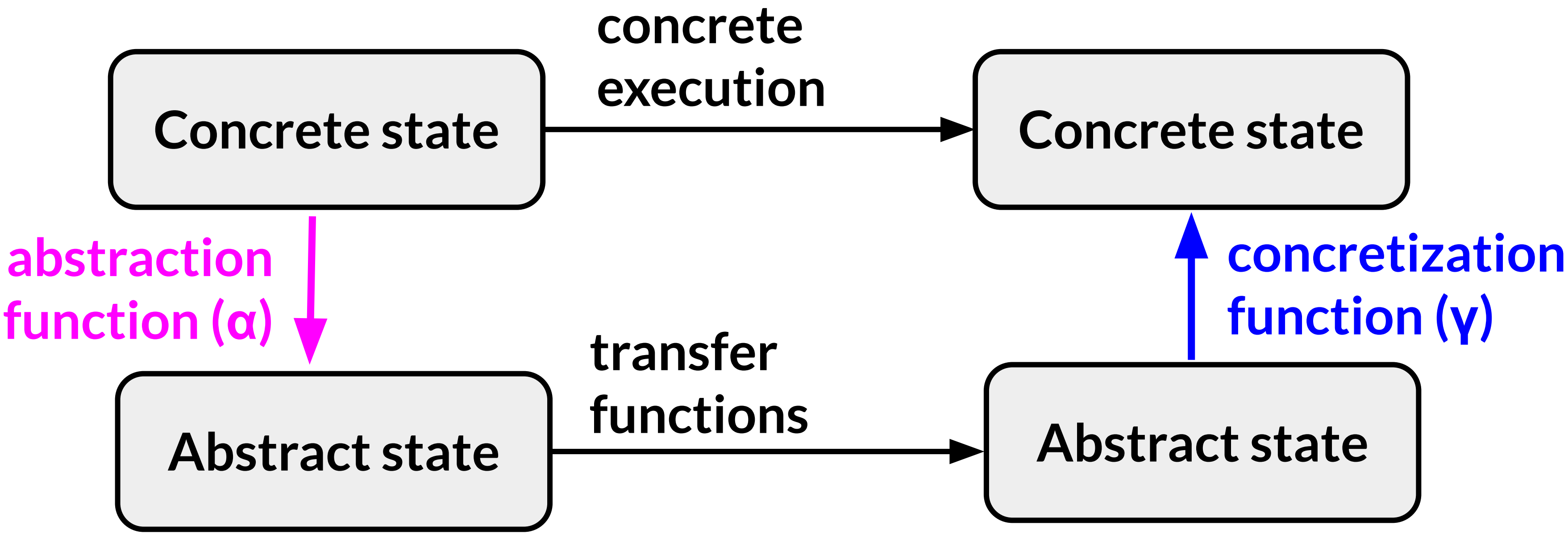

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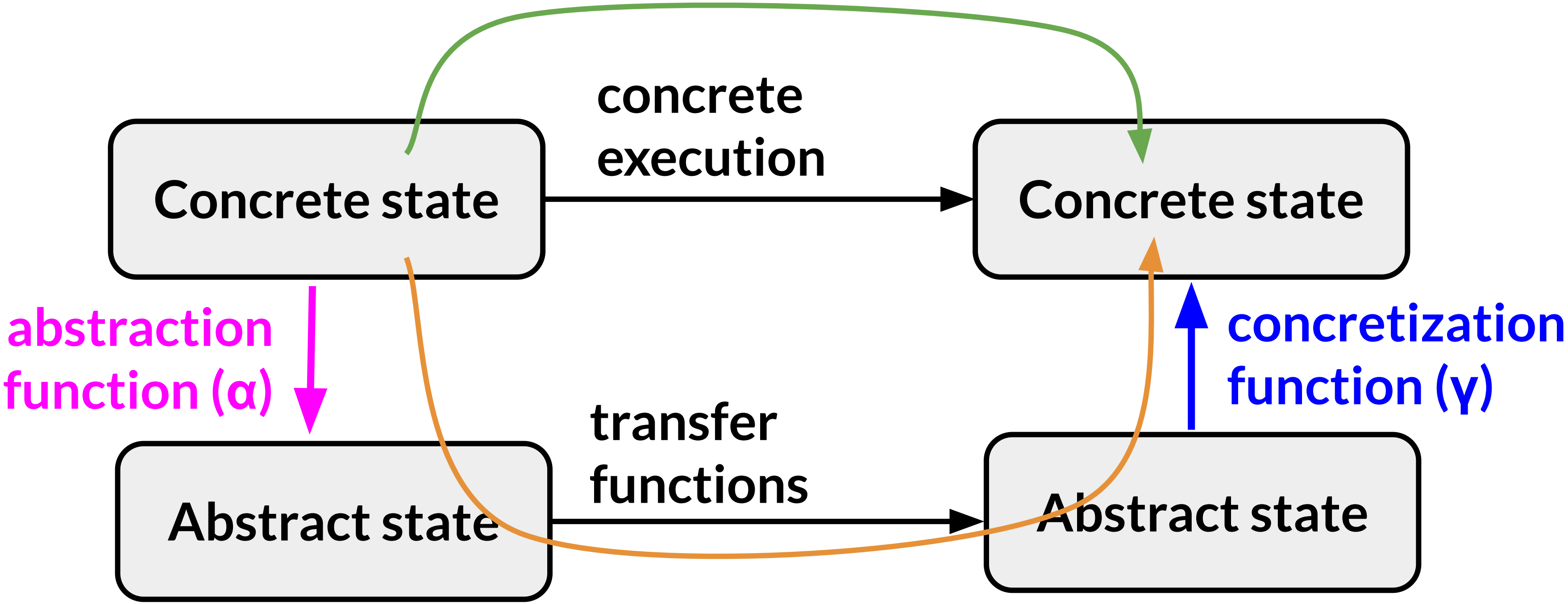

soundness means that the **green path** is a subset of the **orange path**

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- last week, I went through an extended example of how to get a parity analysis to work on **one program**
  - however, that was just an example!
    - an abstract interpretation is applicable to **any program**
  - one of the **key challenges** in abstract interpretation design is figuring out the **right set of examples** to handle precisely

Control Flow Graphs

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- this is the internal representation used by most static analysis tools
- nodes in the CFG are ***basic blocks***
  - a basic block is a sequence of instructions that always execute together

Control Flow Graphs: Example

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x <- 0 ;
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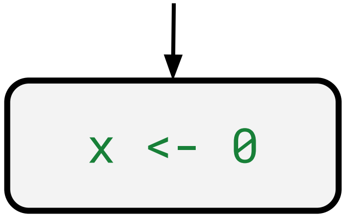

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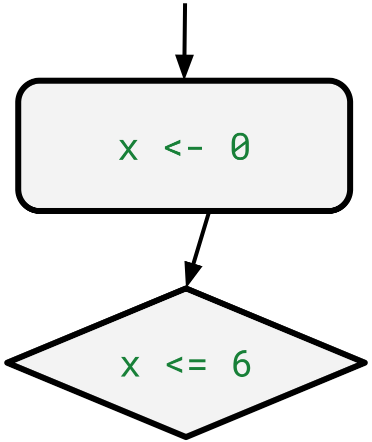

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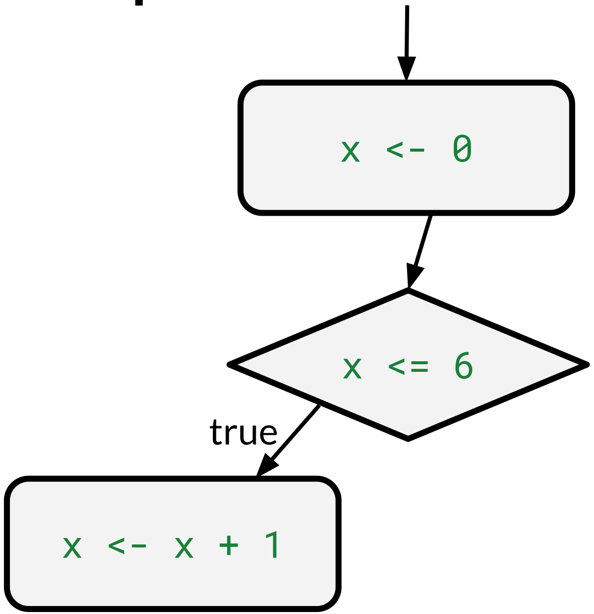

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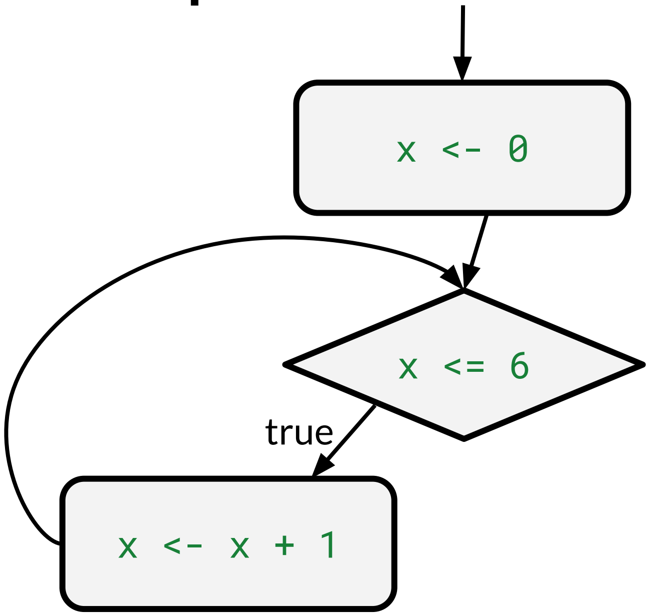

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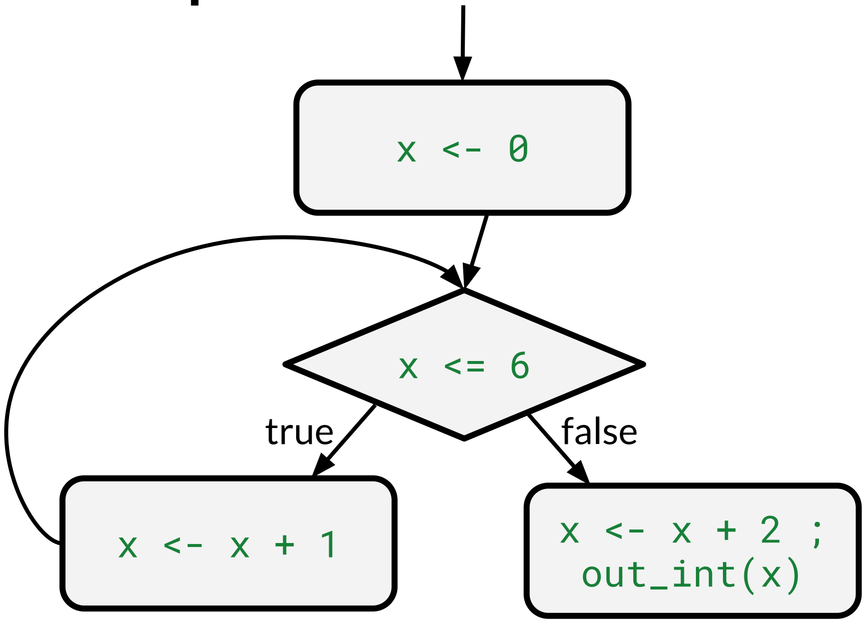

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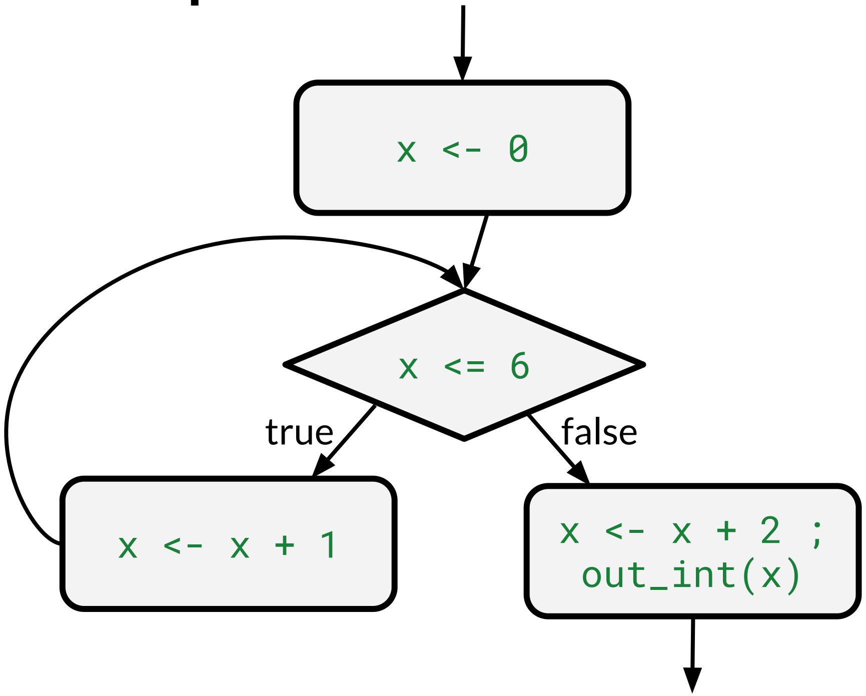

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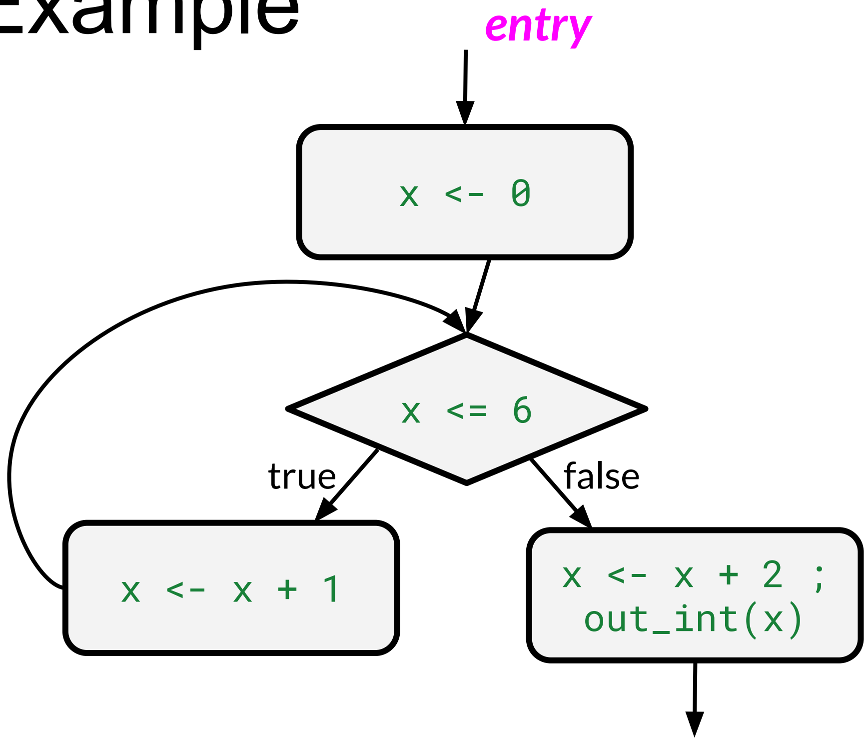

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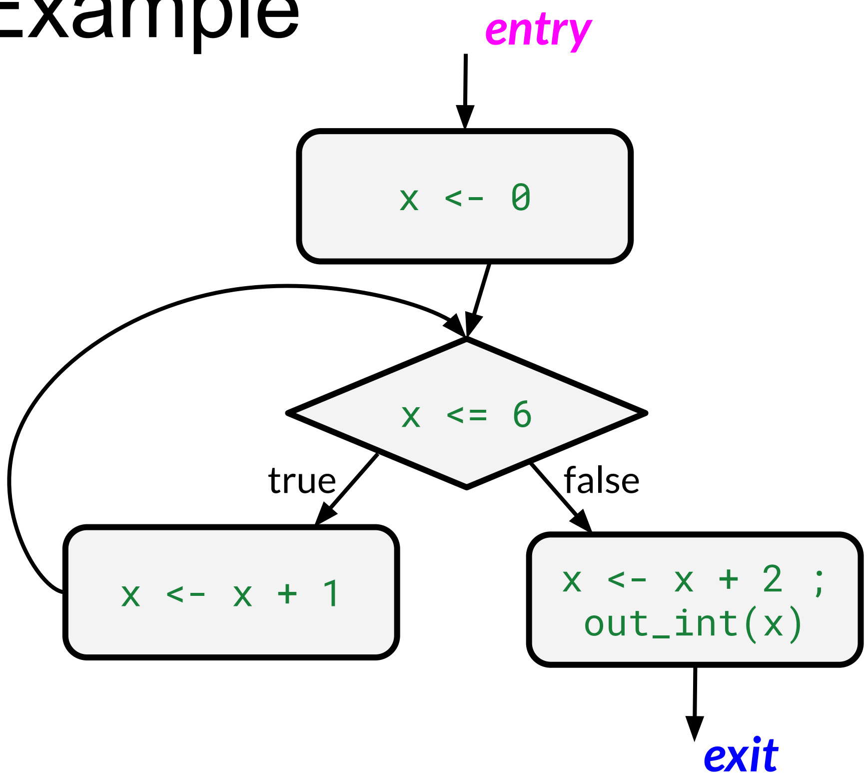

Agenda: abstract interpretation, part 2

- review and clarifications from last week
- **soundness**
- refinement and branching
- widening
- Stein’s algorithm example

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  - but this is too strong: approximation may cause us to lose information! So, the standard formalism is:
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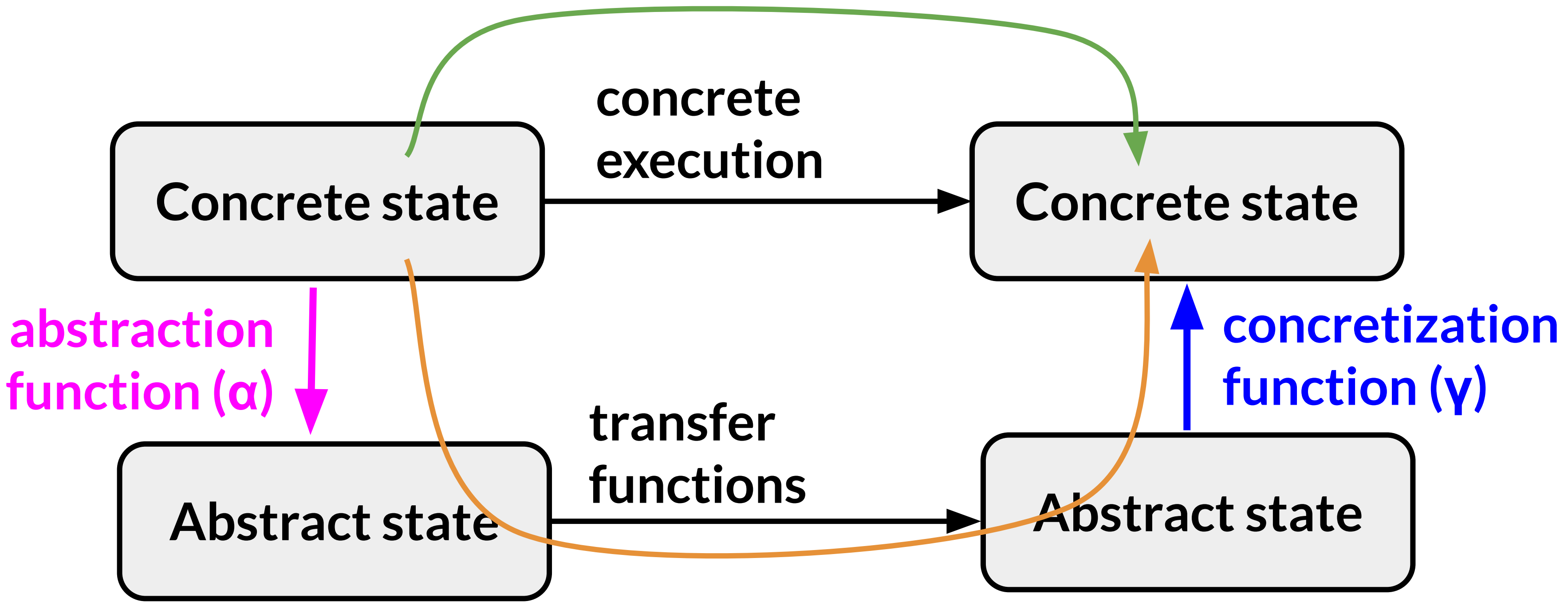
And, it's also necessary to show that the Galois connection holds for the **transfer functions**!

Approximation!

Remember this diagram from earlier?



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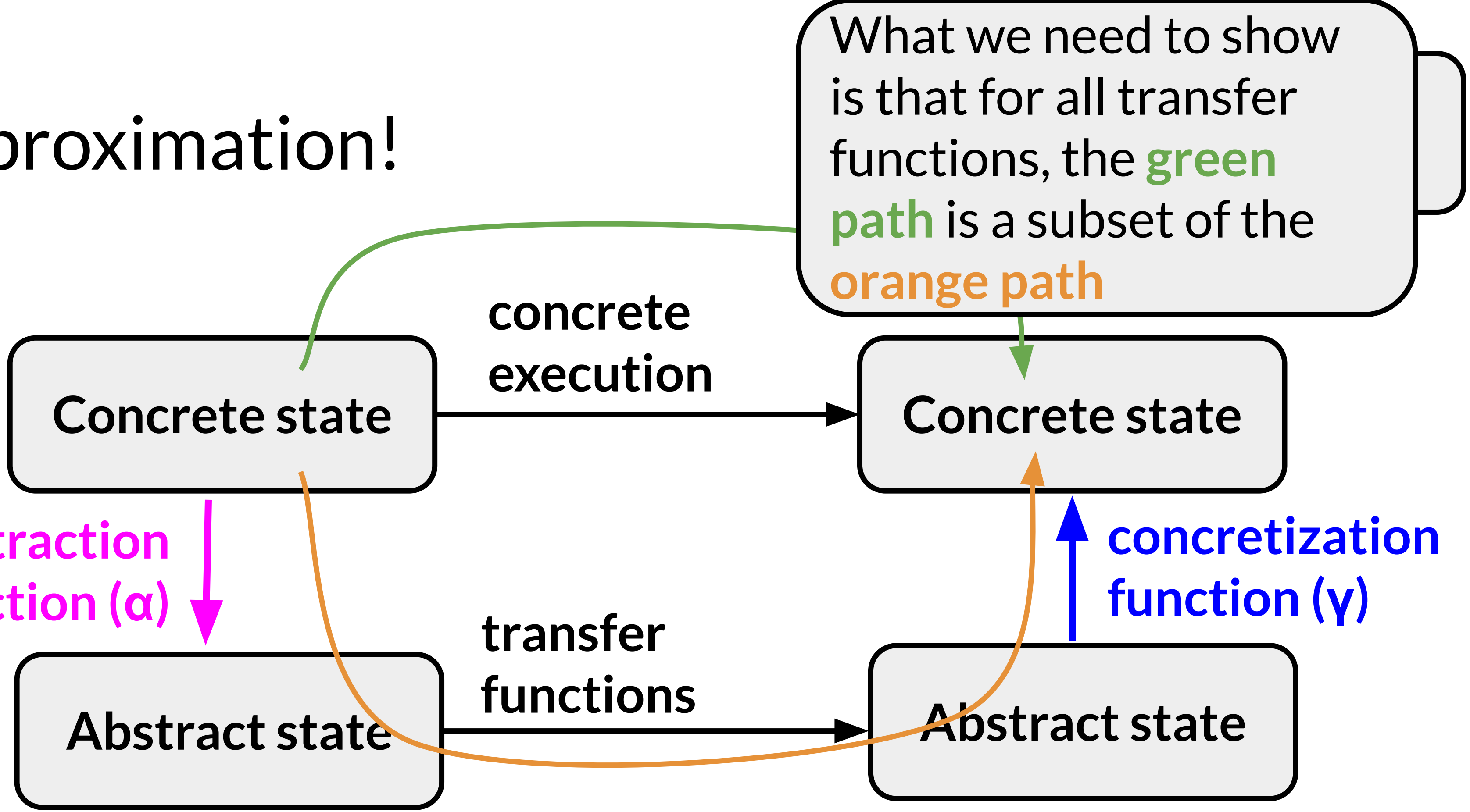

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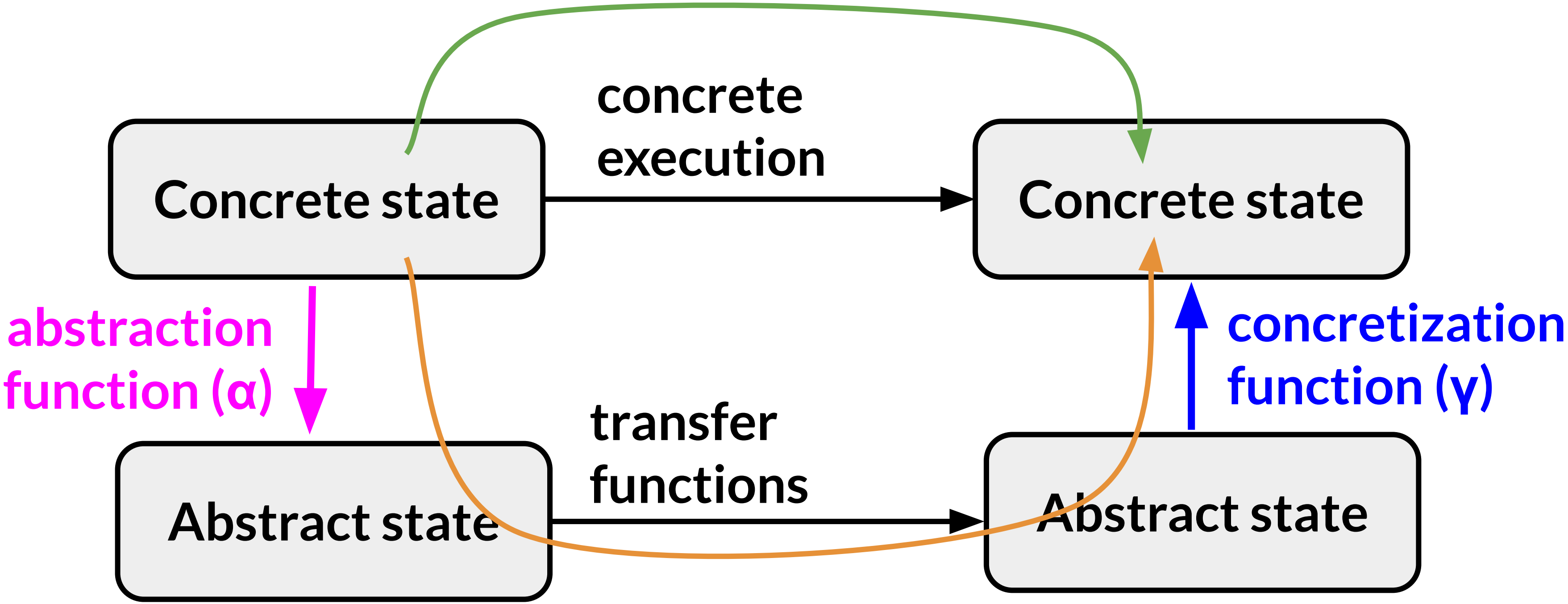
More on soundness: using a Galois connection



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possible results of concrete execution (**green line**)

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concretization of the result of applying the **transfer function** to the **abstraction** of the original concrete state (orange line)

More on soundness: example proof

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      /          \
  {even}      {odd}
      \          /
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T	T	T	T	⊥
even	T	even	odd	⊥
odd	T	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

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even	⊤	even	odd	⊥
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    - {} ⊆ {}

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    - {} $\subseteq$ {}
    - QED

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T	∓	∓	∓	⊥
even	∓	even	odd	⊥
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⊥	⊥	⊥	⊥	⊥

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even	∓	even	odd	±
odd	∓	odd	even	±
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  - but w… is commutative
    - so… rder

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even	⊤	even	**odd**	⊥
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in other words, the two **orange** cases are the same!

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Agenda: abstract interpretation, part 2

- review and clarifications from last week
- soundness
- **refinement and branching**
- widening
- Stein’s algorithm example

Refinement

Consider the following program:

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Insight: *anything* you can figure out by reasoning through the program by hand, an abstract interpretation can do too!

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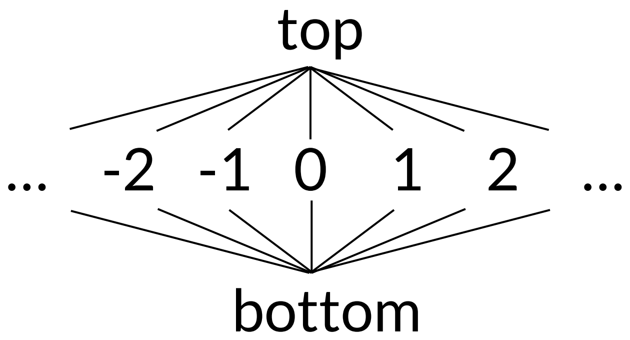

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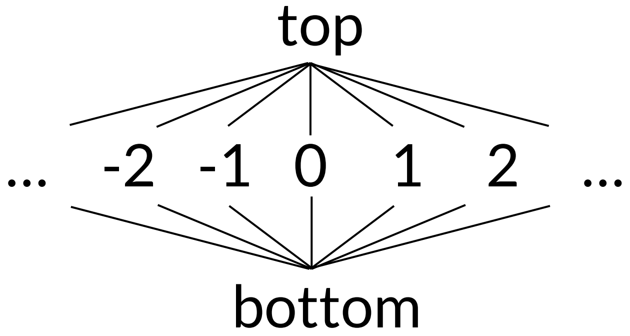

not enough! need sets

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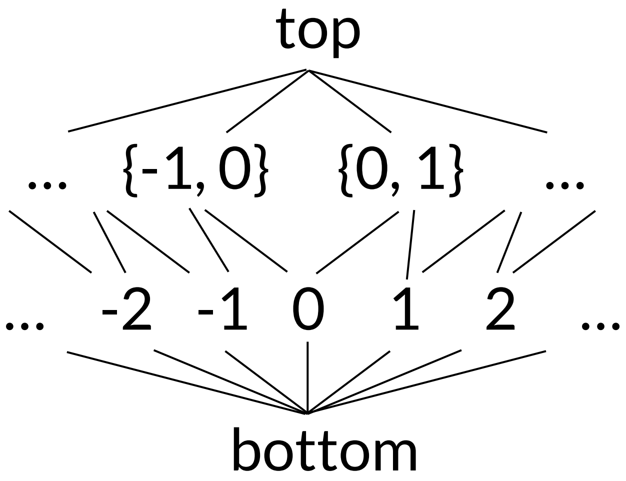

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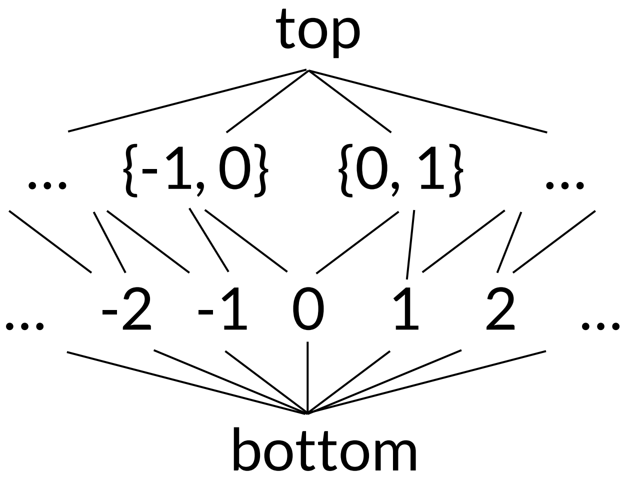

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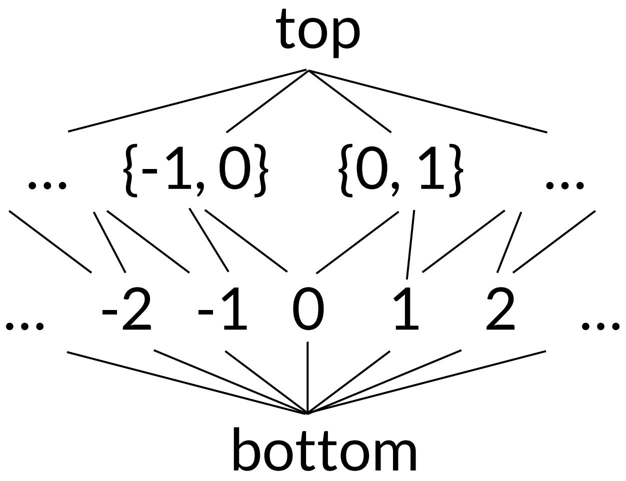
Does this permit us to prove the value of x at the end?
NO (need transfer function)



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  - why >= and not < ?
    - loop guard is false, so we invert the operator

Refinement

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(on the whiteboard. Start by drawing a CFG, then execute the algorithm. Put the CFG to the side and don't erase it.)

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  - do you think that's possible?
    - We can use ***widening operators*** to allow this (sort of)

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- this is safe because the analysis isn't required to take **the least** upper bound so long as it chooses **an** upper bound
- example widening operator for constant propagation:
  - if an abstract value has changed at least five times, go to top

Widening

Let’s return to the previous example:

```
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  - but abstract interpretation was always imprecise, so that's okay
- A nice fact about implementing an abstract interpretation is that it is **always safe** to apply a widening operator
  - this means it's easy to support complex language features: just immediately widen any values that they impact
    - "go to top" is a sound policy in all situations

Agenda: abstract interpretation, part 2

- review and clarifications from last week
- soundness
- refinement and branching
- widening
- **Stein's algorithm example**

Another example: Stein's algorithm

```
def gcd(int a, int b):
  if a == 0 or b == 0:
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  int expt = 0
  while a is even and b is even:
    a = a / 2
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    expt = expt + 1
  while a != b:
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  - 2 != 0
- "constant propagation" can prove no divisions by zero!

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- 1st: a is odd or b is odd
- 2nd: a eventually equals b

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Fortunately, we already know an analysis for parity. Let's use it (on the board; requires a CFG).

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- we ran into a problem: we **can't prove** that a and b are eventually odd!
  - the transfer function for even / is2 returns T
- in this case, that's actually correct!
  - the program does not terminate on all inputs
  - -1, 1 is a counterexample

Course Announcements

- PA2 due today!
- PA3c1 (codegen testing) is due on Friday
  - all gas, no brakes
- My OH on Wednesday will be later than usual (4-5 instead of 3:30-4:30), because of a CS faculty meeting until 4
  - might even start a little bit later…