Abstract Interpretation (1/2)

Martin Kellogg

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- Today: definitions, examples, soundness (?)
- Next class: more theory and examples

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When dealing with a concrete language, we don't usually get to **choose** the domain or the semantics. But in abstract interpretation, we do!

Domains

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  - an ***abstract domain*** is a layer of indirection on top of the concrete domain that splits the concrete domain into a smaller number of sets

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  - **prime/composite**
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  - many more!

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Important property of an abstract domain: it must **completely cover** the concrete domain

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  - each $\boldsymbol{A_i}$ represents an ***abstract value***
    - e.g., “odd integers”, “Strings that match my regular expression”, etc.

Domains: orderings and lattices

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- There are two ways to express the ordering:
  - define a **less than relation** (usually denoted by $\sqsubset$), or
  - define a **least upper bound operator** (usually denoted by $\sqcup$)
- These two approaches are **equivalent**: you can derive the LUB from the less than relation and vice-versa

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    - formally, we define a relation as a set of ordered pairs
- If $x \sqsubset y$, then we say that x is lower or less, and that y is higher or greater
- The less-than relation **need not be total**
    - for two points $e1$ and $e2$, it is possible that neither $e1 \sqsubset e2$ nor $e2 \sqsubset e1$ is true

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- The least upper bound is often more useful, because it directly models the **join operator**
  - that is, it models what happens when two possible abstract values flow to the same location (e.g., the then and else branches of an if)

Least upper bound: relationship to types

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              /        \
       Animal            Shape
      /      |             |   \
  Bird    Mammal       Circle  Rect
           /   \                 |
        Dog    Cat            Square
```

Least upper bound: relationship to types

- You are already familiar with the LUB operator from our discussion of type systems and your experience with **object-oriented programming**
    - any time that you've answered the question "what is the closest supertype that these two types share", you're doing a LUB

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    - LUB is a binary function; for a binary function f, monotonicity is defined as
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    - Note that this is not the same as:
      - $\forall$ x, y . f(x, y) $\sqsupseteq$ x $\land$ f(x, y) $\sqsupseteq$ y!
      - though this property is also true of the LUB operator

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Hint: I like to ask exam questions like "why is this property required?" or "what would happen if it weren't true?"

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A set is ***partially ordered*** iff $\exists$ a binary relationship $\leq$ that is:

- ***reflexive***: $x \leq x$
- ***anti-symmetric***: $x \leq y \wedge y \leq x \Rightarrow x = y$
- ***transitive***: $x \leq y \wedge y \leq z \Rightarrow x \leq z$

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A note about top:

- top represents ***no constraints*** on the possible values
- equivalently, ***every value*** is a member of top

Example AI: even/odd integers

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Similarly for bottom:

- bottom represents ***all possible constraints at once*** on values
- equivalently, ***no values*** are members of bottom

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Example transfer function:

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odd	T	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

Example AI: even/odd integers

Let’s apply this AI to an example:

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x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
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Concrete execution

```
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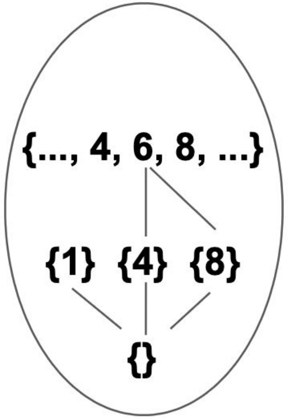

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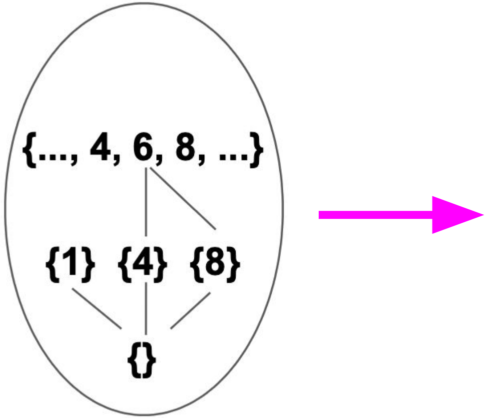

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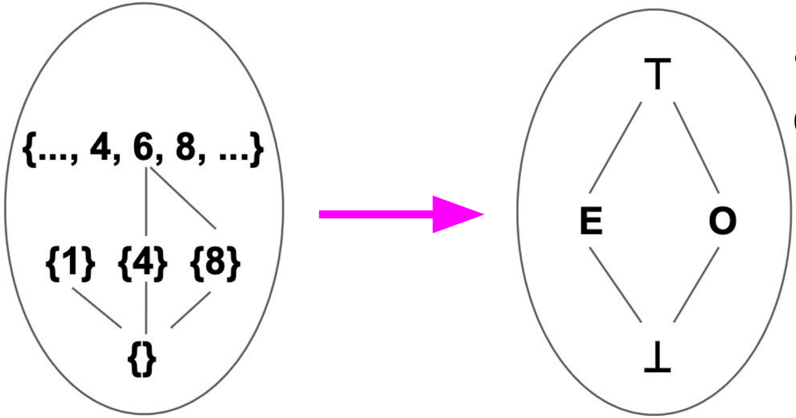

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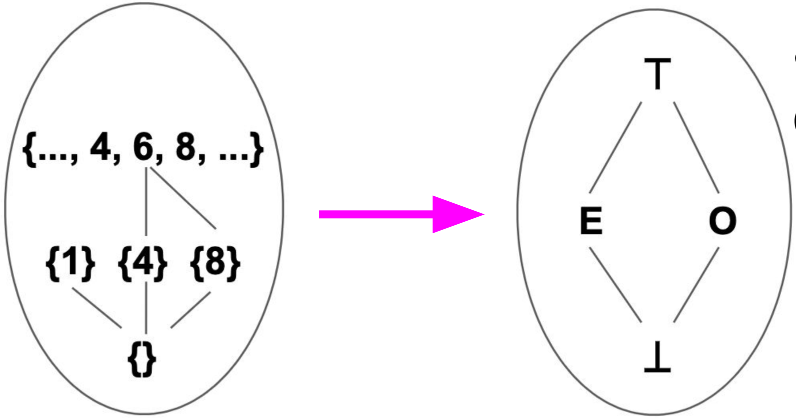

abstract domain

e.g.:
α(4) = even
α({}) = bottom

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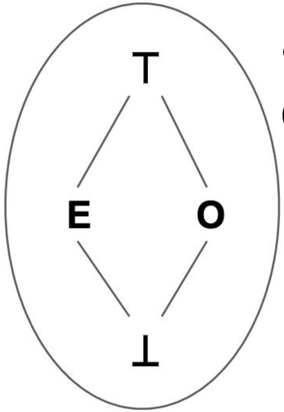

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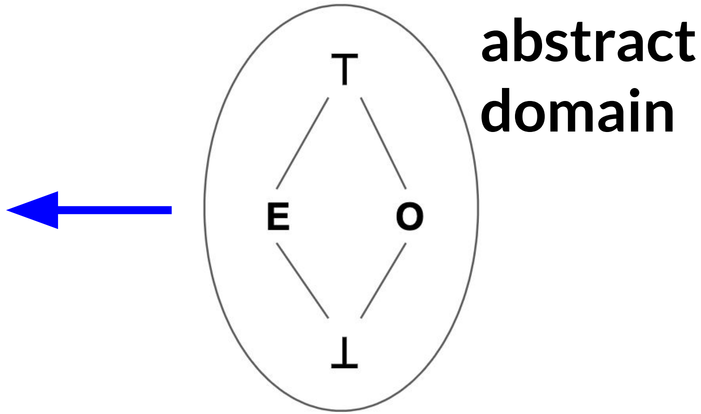

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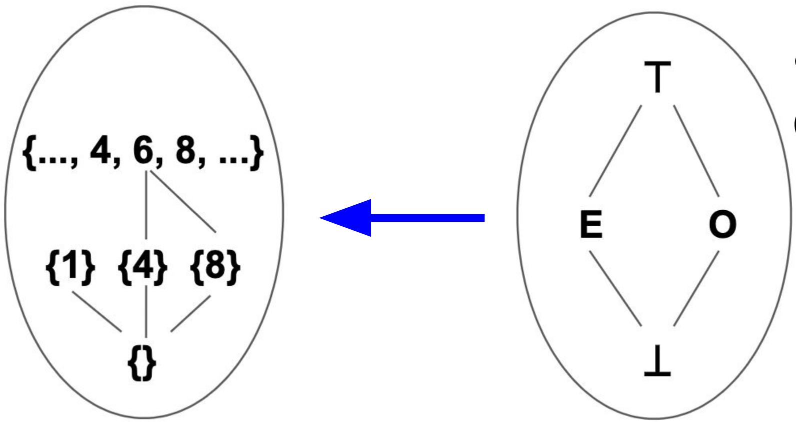

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Concrete state

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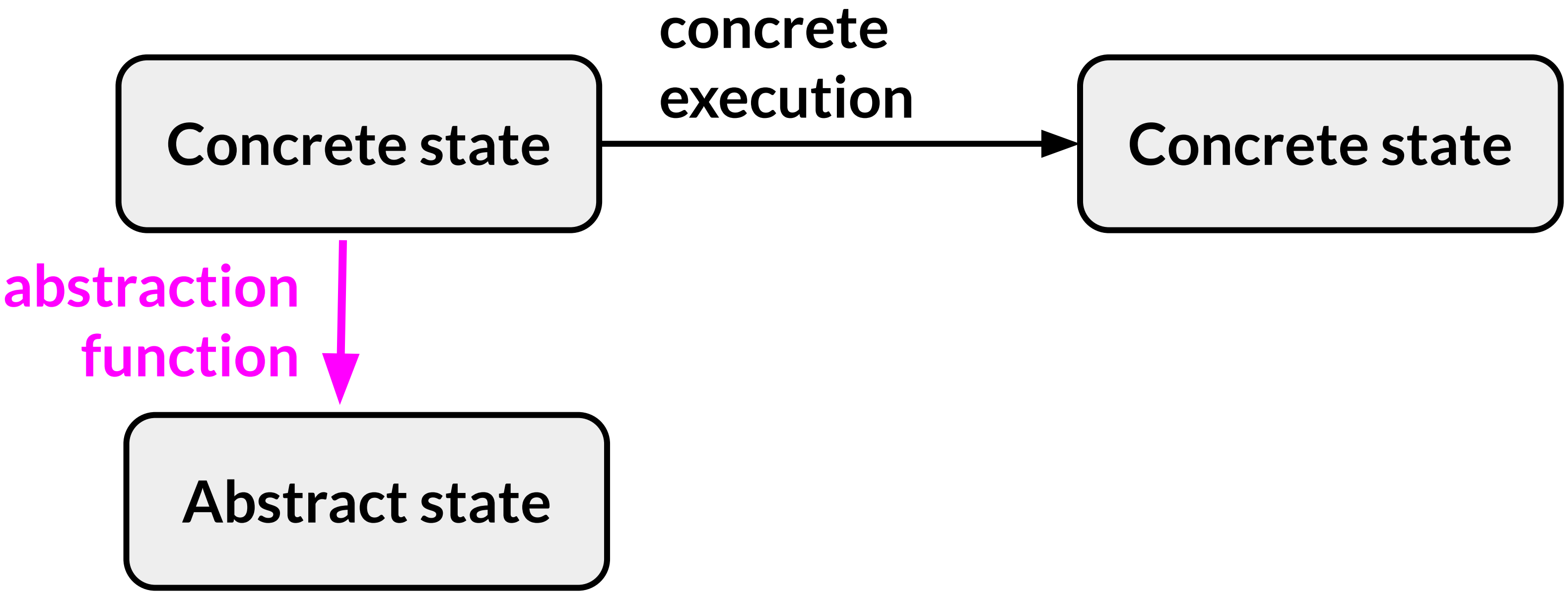
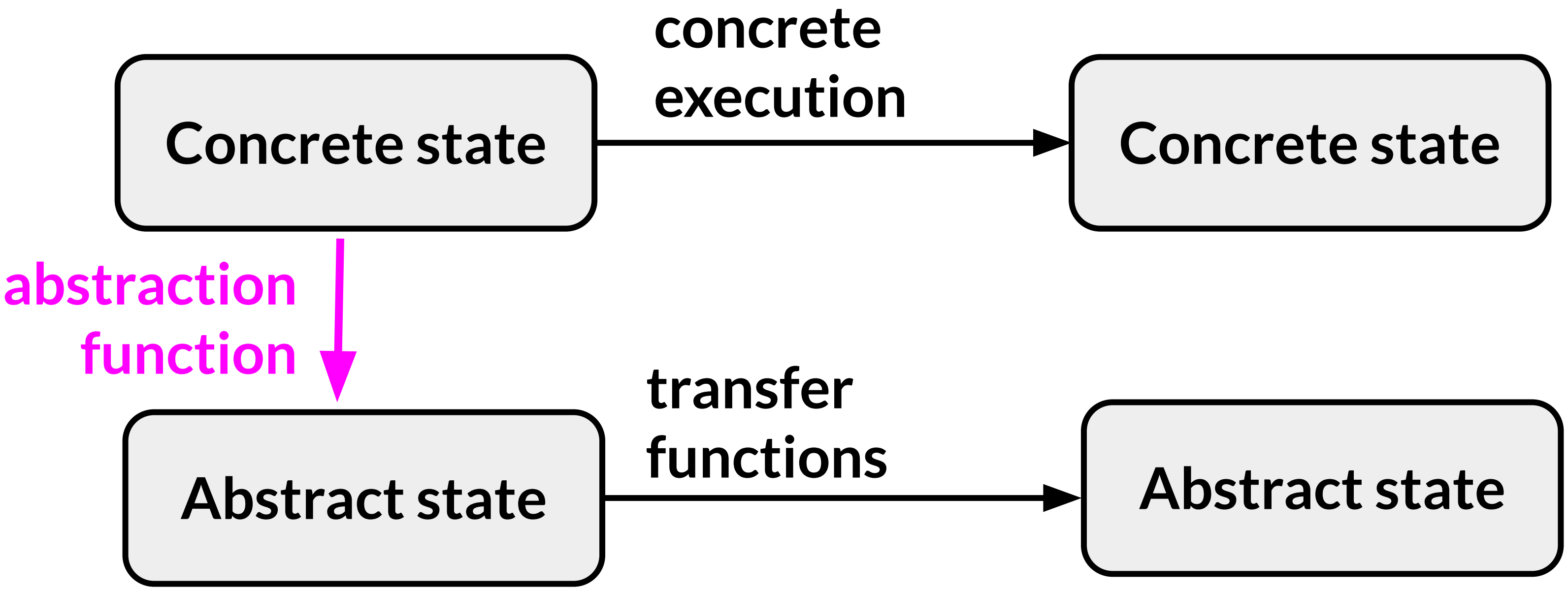
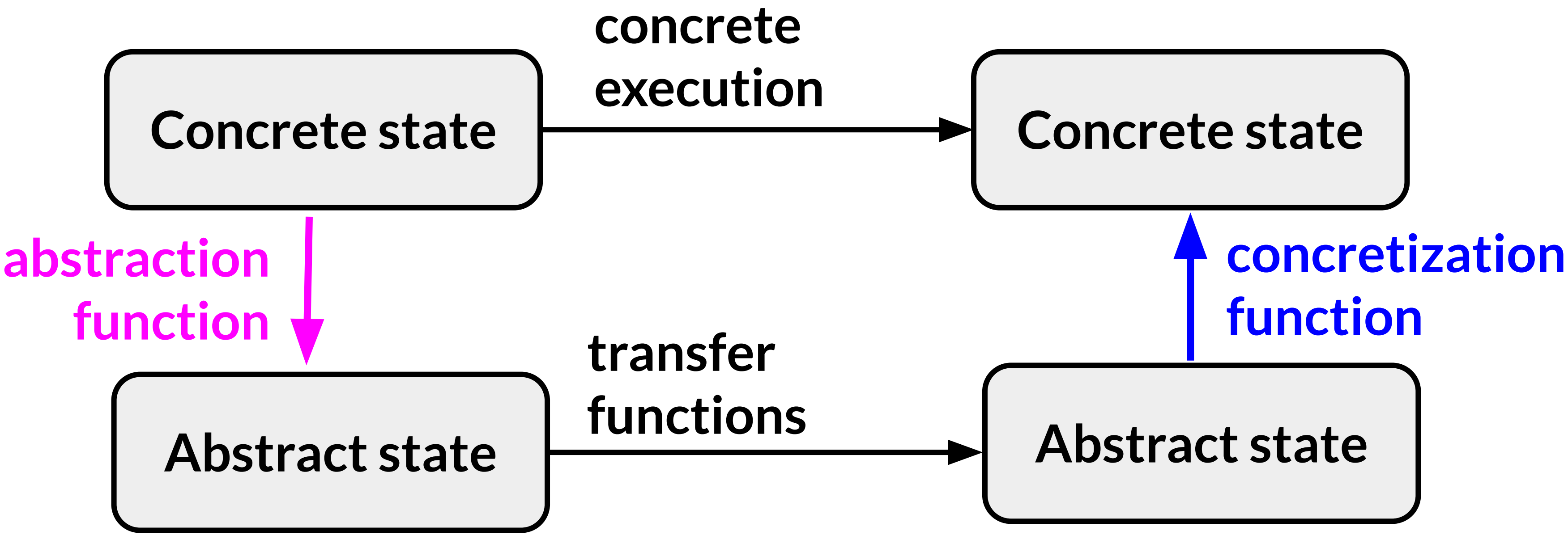


Role of abstr., concr., and transfer fcns.



Role of abstr., concr., and transfer fcns.



Example AI: even/odd integers

Let's apply this AI to an example:

```
x = 0;
y = read_even()
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Concrete execution	Abstract interpr.
{x=0; y=undef}	{x=**e**; y=?}
{x=0; y=8}	{x=?; y=?}
{x=9; y=8}	{x=?; y=?}
{x=9; y=18}	{x=?; y=?}
{x=16; y=18}	{x=?; y=?}
{x=16; y=8}	{x=?; y=?}

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x = 0;
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Concrete execution		Abstract interpr.	
{x=0;	y=undef}	{x=**e**;	y=$\perp$}
{x=0;	y=8}	{x=?;	y=?}
{x=9;	y=8}	{x=?;	y=?}
{x=9;	y=18}	{x=?;	y=?}
{x=16;	y=18}	{x=?;	y=?}
{x=16;	y=8}	{x=?;	y=?}

Example AI: even/odd integers

Let's apply this AI to an example:

Code	Concrete execution	Abstract interpr.
`x = 0;`	`{x=0; y=undef}`	{x=**e**; y=⊥}
`y = read_even()`	`{x=0; y=8}`	{x=**e**; y=**e**}
`x = y + 1;`	`{x=9; y=8}`	{x=?; y=?}
`y = 2 * x;`	`{x=9; y=18}`	{x=?; y=?}
`x = y - 2;`	`{x=16; y=18}`	{x=?; y=?}
`y = x / 2;`	`{x=16; y=8}`	{x=?; y=?}

Example AI: even/odd integers

Let's apply this AI to an example:

transfer function for +!

```
x = 0;
y = read_even()
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Concrete execution	**Abstract interpr.**
{x=0; y=undef}	{x=**e**; y=$\perp$}
{x=0; y=8}	{x=**e**; y=**e**}
{x=9; y=8}	{x=**o**; y=**e**}
{x=9; y=18}	{x=?; y=?}
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Concrete execution	Abstract interpr.
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{x=9; y=18}	{x=o; y=e}
{x=16; y=18}	{x=?; y=?}
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{x=9;	y=8}	{x=**o**;	y=**e**}
{x=9;	y=18}	{x=**o**;	y=**e**}
{x=16;	y=18}	{x=**e**;	y=**e**}
{x=16;	y=8}	{x=?;	y=?}

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{x=9;	y=18}	{x=**o**;	y=**e**}
{x=16;	y=18}	{x=**e**;	y=**e**}
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Example AI: even/odd integers

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{x=9;	y=18}	{x=**o**;	y=**e**}
{x=16;	y=18}	{x=**e**;	y=**e**}
{x=16;	y=8}	{x=**e**;	y=**e**?}

Example AI: even/odd integers

What's the transfer function for division?

↓/→	⊤	even	odd	⊥
⊤				
even				
odd				
⊥				

Example AI: even/odd integers

What’s the transfer function for division?

↓/→	⊤	even	odd	⊥
⊤	⊤	⊤	⊤	⊥
even	⊤	⊤	⊤	⊥
odd	⊤	⊤	⊤	⊥
⊥	⊥	⊥	⊥	⊥

Notes for online readers:

- even/even is top:
  - 6/2 = 3
  - 8/2 = 4
- odd/odd is top:
  - 5/5 = 1
  - 11/5 = 2
    - integer division!

Example AI: even/odd integers

Let's apply this AI to an example:

```
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{x=9; y=18}	{x=**o**; y=**e**}
{x=16; y=18}	{x=**e**; y=**e**}
{x=16; y=8}	{x=**e**; y=**T**}

Example AI: even/odd integers

Let's apply this AI to an example:

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{x=9;	y=18}	{x=**o**;	y=**e**}
{x=16;	y=18}	{x=**e**;	y=**e**}
{x=16;	y=8}	{x=**e**;	y=**T**}

for x, our abstraction was precise

Example AI: even/odd integers

Let’s apply this AI to an example:

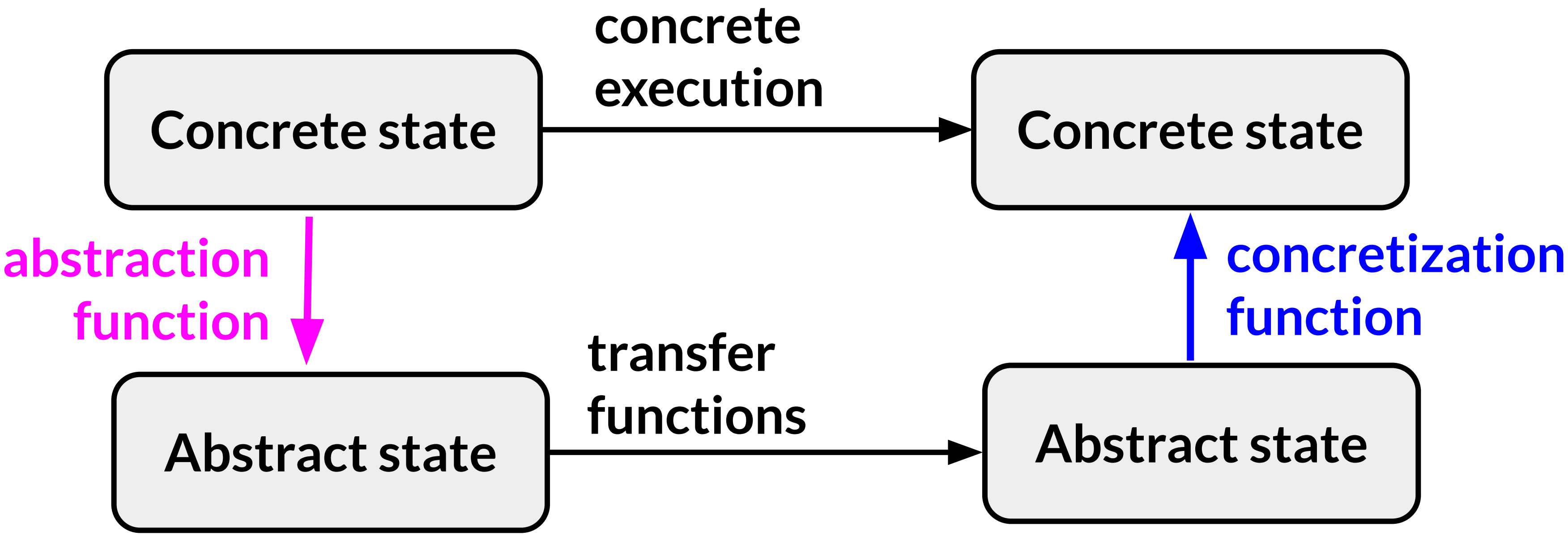
```
x = 0;
y = read_even()
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Concrete execution		**Abstract interpr.**	
{x=0;	y=undef}	{x=**e**;	y=$\bot$}
{x=0;	y=8}	{x=**e**;	y=**e**}
{x=9;	y=8}	{x=**o**;	y=**e**}
{x=9;	y=18}	{x=**o**;	y=**e**}
{x=16;	y=18}	{x=**e**;	y=**e**}
{x=16;	y=8}	{x=**e**;	y=**T**}

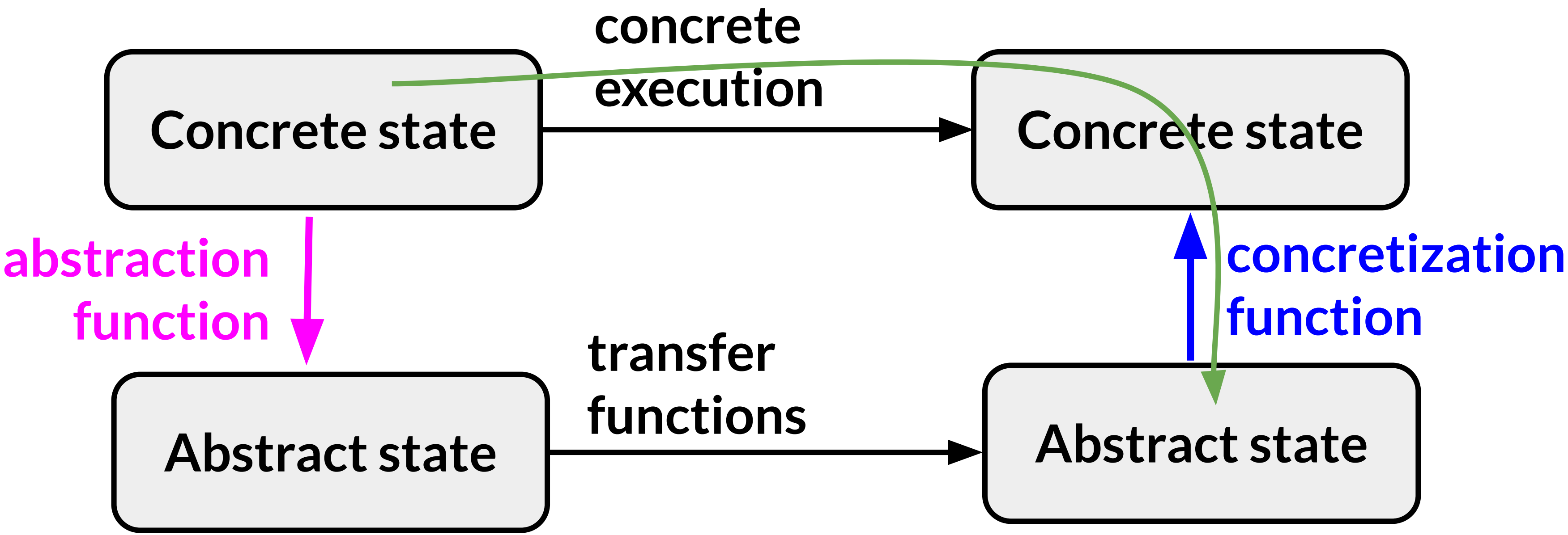
for x, our abstraction was precise

but for y, it was not

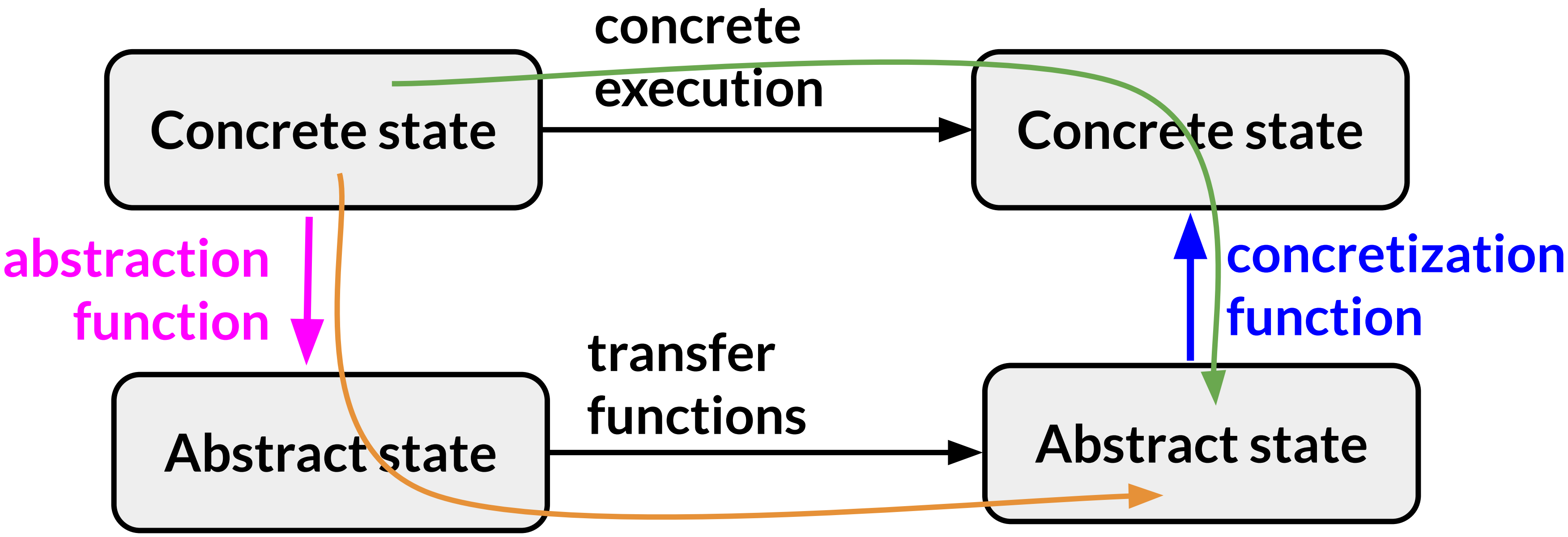
Approximation!



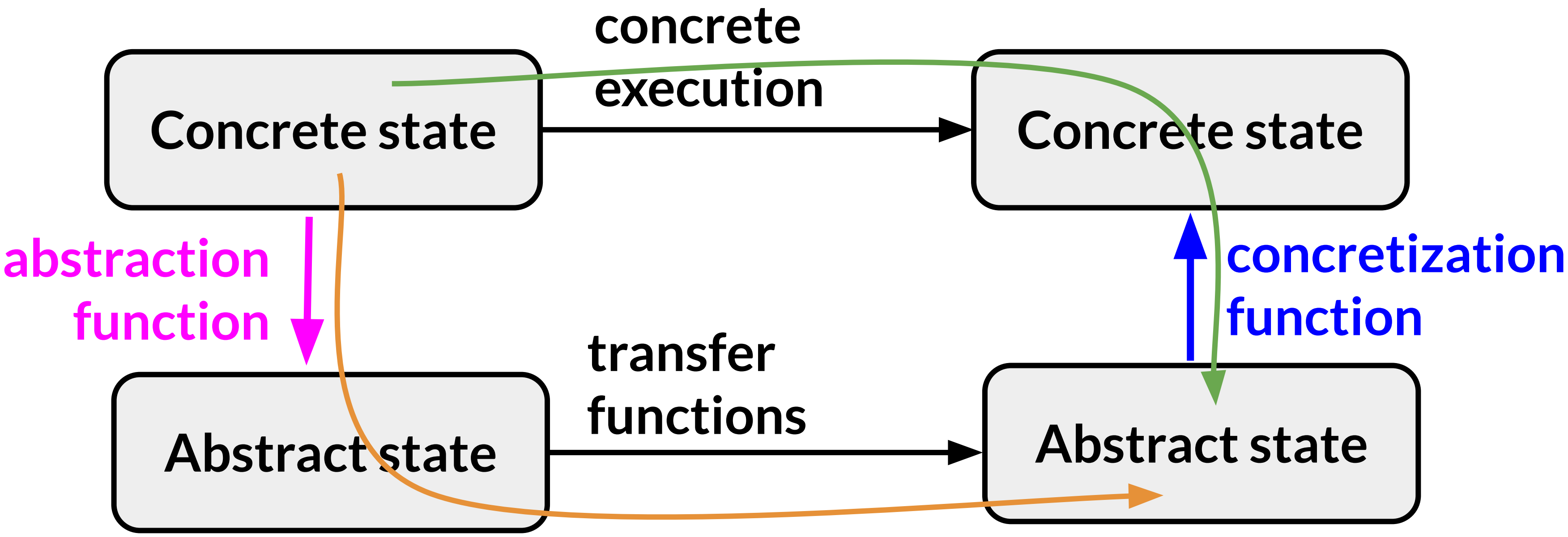
Approximation!



Approximation!

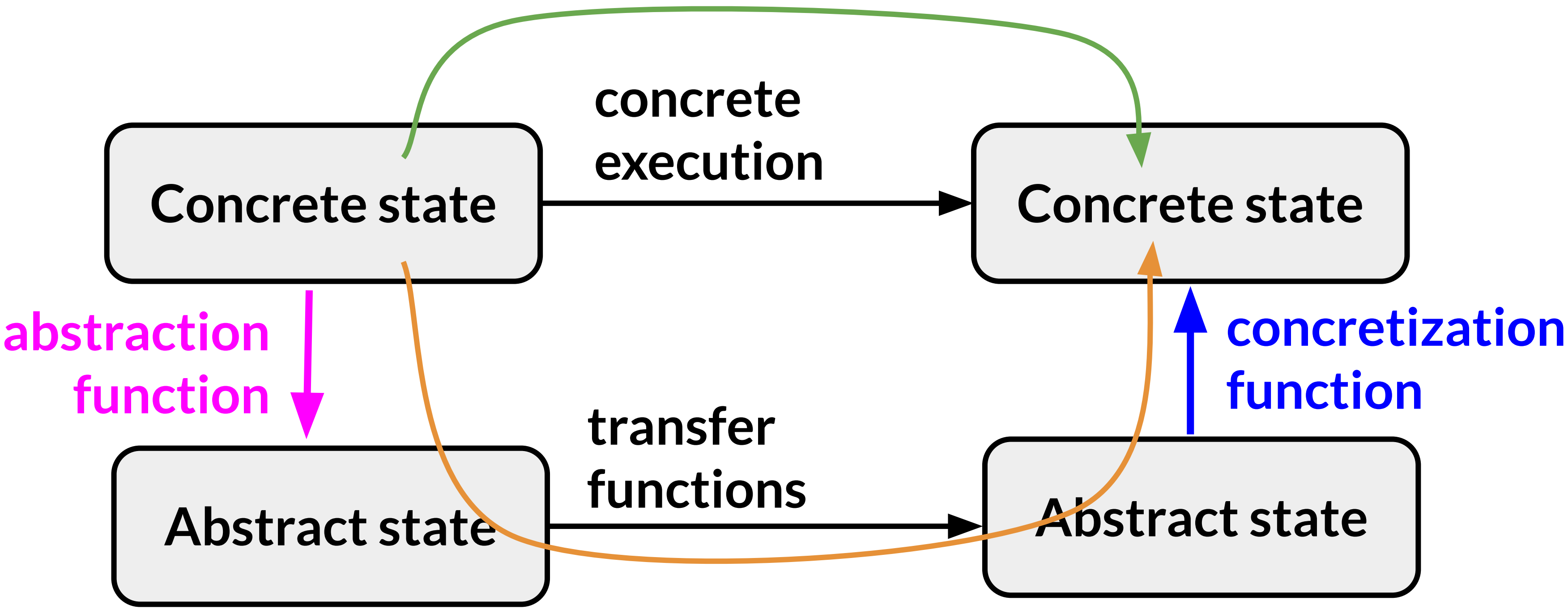


Approximation!



Do the green and orange paths always lead to the same abstract state?

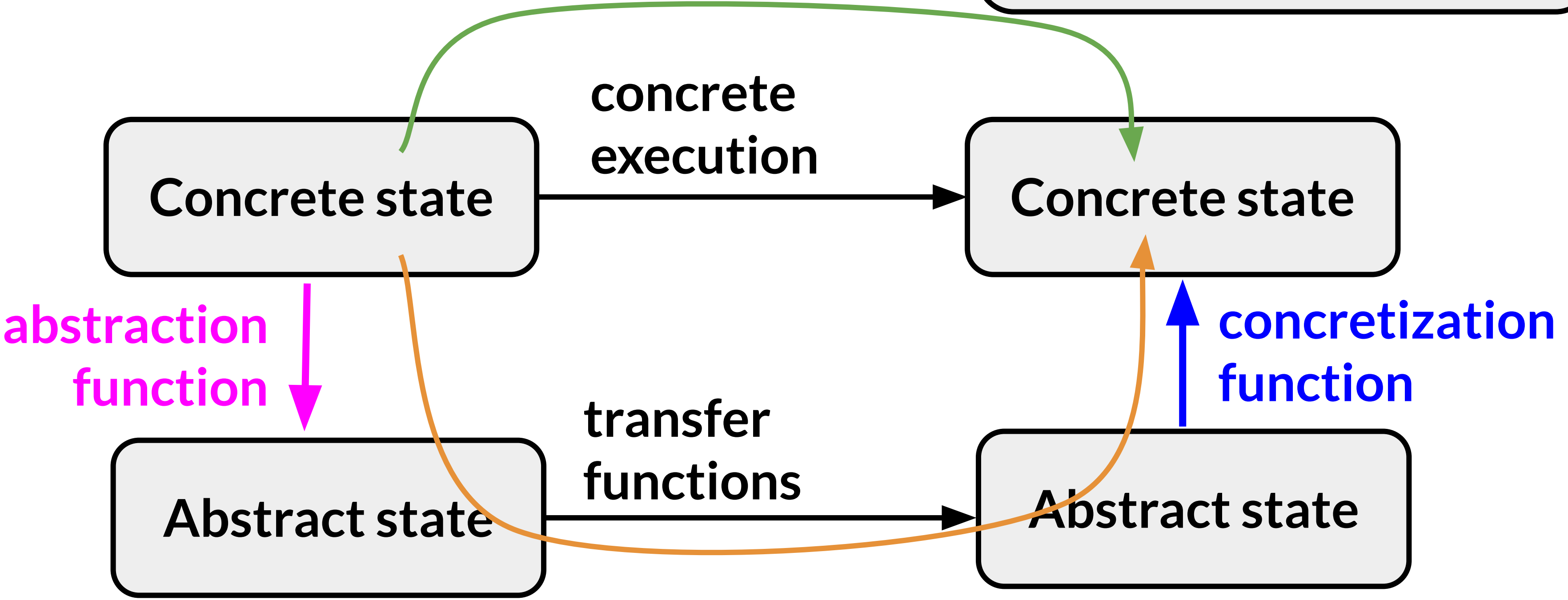
Approximation!



Do the green and orange paths always lead to the same concrete state?

Approximation!

We'll come back to this question when we discuss **soundness**



Do the green and orange paths always lead to the same concrete state?

Trivia Break: Building Materials

This material was in widespread use by 150 BCE; some scholars believe that it was developed at least a century earlier. Its widespread use enabled the construction of a number of architecturally-innovative buildings, including the Pantheon’s dome (built 113-125 CE), which is still the largest unreinforced dome of this material in the world. Unlike its modern equivalent, it was laid rather than poured. Incorporation of different types of lime enabled this material to “self-repair” cracks, contributing to its longevity.

Trivia Break: Computer Science

This family of multi-tasking, multi-user computer operating systems is distinguished from its predecessors by being the first "portable" operating system (i.e., it could run on more than one model of computer). Development of its first version began in 1969. It is characterized by an eponymous design philosophy that argues that an operating system should provide a set of simple tools, each of which performs a limited, well-defined function; and that larger programs should be built by composing these tools.

Name the family of operating systems and the place where the first version of this operating system was created.

Alternative example AI: even/odd integers

Is there an **alternative** AI that we can use to conclude that y is even after we analyze the example?

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Alternative example AI: even/odd integers

Is there an alternative AI that we can use to conclude that y is even after we analyze the example?

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

In-class exercise: with a partner, ***design an alternative*** abstract interpretation that can conclude that y is even.

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

- ask yourself: "**for what x is that true?**"

Alternative example AI: even/odd integers

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- ask yourself: “**for what x is that true?**”
  - simplest answer: `x . x%4 = 0` - that is, all `x`s such that x is **divisible by 4**

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

- ask yourself: "**for what x is that true?**"
  - simplest answer: `x.x%4 = 0` - that is, all `x`s such that x is **divisible by 4**
  - alternative answer: abstract value tracks the number of 2s in the prime factorization

Alternative example AI: even/odd integers

Key property that we need to conclude is that `x / 2` is even.

- ask yourself: "**for what x is that true?**"
  - simplest answer: `x . x%4 = 0` - that is, all `x`s such that x is **divisible by 4**
  - alternative answer: abstract value tracks the number of 2s in the prime factorization
- cunning plan: add a "divisible by 4" abstract value (**mod4**) to our lattice, then rebuild our transfer functions

Alternative example AI: even/odd integers

Next question: where does “divisible by 4” go in the **lattice**?

```
{ even, odd } = top
     /         \
  {even}     {odd}
     \         /
     {} = bottom
```

Alternative example AI: even/odd integers

Next question: where does “divisible by 4” go in the **lattice**?

```
{ even, odd } = top
     /          \
  {even}      {odd}
     |           |
  {mod4}         |
     \          /
      {} = bottom
```

all mod4 integers are also even!

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

recall our **original** transfer function for +:

+	T	even	odd	⊥
T	T	T	T	⊥
even	T	even	odd	⊥
odd	T	odd	even	⊥
⊥	⊥	⊥	⊥	⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

recall our **original** transfer function for +:

we need to add a row and a column for **mod4**:

+	T	even	odd	mod4	⊥
T	T	T	T		⊥
even	T	even	odd		⊥
odd	T	odd	even		⊥
mod4					
⊥	⊥	⊥	⊥		⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

recall our **original** transfer function for +:

we need to add a row and a column for **mod4**:

+	$\top$	even	odd	mod4	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
even	$\top$	even	odd	even	$\bot$
odd	$\top$	odd	even	odd	$\bot$
mod4	$\top$	even	odd	mod4	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

same thing for **division**:

↓/→	⊤	even	odd	mod4	⊥
⊤	⊤	⊤	⊤		⊥
even	⊤	⊤	⊤		⊥
odd	⊤	⊤	⊤		⊥
mod4					
⊥	⊥	⊥	⊥		⊥

Alternative example AI: even/odd integers

How to change our **transfer functions**? Let's do two examples (+ and /):

same thing for **division**:

oh no! why is mod4 divided by even top?

- 4/4 = 1 :(
- we need **another** lattice element to make this work!

$\downarrow/\rightarrow$	$\top$	even	odd	mod4	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\bot$
even	$\top$	$\top$	$\top$	$\top$	$\bot$
odd	$\top$	$\top$	$\top$	$\top$	$\bot$
mod4	$\top$	$\top$	$\top$	$\top$	$\bot$
$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$

Alternative example AI: even/odd integers

Another lattice element: "**is2**"

Alternative example AI: even/odd integers

Another lattice element: "**is2**"

- **sibling** of mod4 in the lattice

Alternative example AI: even/odd integers

Another lattice element: "**is2**"

- **sibling** of mod4 in the lattice

```
      { even, odd } = top
          /           \
     {even}        {odd}
      /      \          |
{mod4}  {is2}      |
       \        |       /
       {} = bottom
```

Alternative example AI: even/odd integers

Another lattice element: “is2”

- sibling of mod4 in the lattice
- its only purpose is to be treated specially in the division transfer function

```
        { even, odd } = top
           /           \
      {even}        {odd}
      /     \          |
{mod4}  {is2}          |
     \      |        /
       {} = bottom
```

Alternative example AI: even/odd integers

Another lattice element: "is2"

- sibling of mod4 in the lattice
- its only purpose is to be treated specially in the division transfer function
  - in particular, we add the rule "mod4 / is2 -> even"
  - full transfer functions left as an exercise

```
      { even, odd } = top
         /          \
   {even}       {odd}
    /     \         |
{mod4} {is2}        |
    \      |       /
     {} = bottom
```

Alternative example AI: let's try it

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Abstract interpr.

```
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
```

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```
x = 0;
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y = x / 2;
```

Abstract interpr.

{x=**e**;	y=⊥}
{x=?;	y=?}
{x=?;	y=?}
{x=?;	y=?}
{x=?;	y=?}
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Abstract interpr.

```
{x=e;    y=⊥}
{x=e;    y=e}
{x=?;    y=?}
{x=?;    y=?}
{x=?;    y=?}
{x=?;    y=?}
```

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Abstract interpr.

{x=**e**;	y=⊥}
{x=**e**;	y=**e**}
{x=**o**;	y=**e**}
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{x=?;	y=?}
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```
x = 0;
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Abstract interpr.

{x=**e**;	y=$\perp$}
{x=**e**;	y=**e**}
{x=**o**;	y=**e**}
{x=**o**;	y=**e**}
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{x=?;	y=?}

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Abstract interpr.

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{x=?;	y=?}

what should the transfer function for even - is2 be?

Alternative example AI: let's try it

```
x = 0;
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y = x / 2;
```

Abstract interpr.

```
{x=e;     y=⊥}
{x=e;     y=e}
{x=o;     y=e}
{x=o;     y=e}
{x=?;     y=?}
{x=?;     y=?}
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what should the transfer function for even - is2 be?

- **even! why not mod4?**

Alternative example AI: let's try it

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```

Abstract interpr.

```
{x=e;    y=⊥}
{x=e;    y=e}
{x=o;    y=e}
{x=o;    y=e}
{x=?;    y=?}
{x=?;    y=?}
```

what should the transfer function for **even - is2** be?

- **even! why not mod4? counterexample: 8 - 2 = 6**

Alternative example AI: let's try it

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Abstract interpr.

{x=**e**;	y=⊥}
{x=**e**;	y=**e**}
{x=**o**;	y=**e**}
{x=**o**;	y=**e**}
{x=**e**;	y=**e**}
{x=?;	y=?}

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Abstract interpr.

{x=**e**;	y=$\perp$}
{x=**e**;	y=**e**}
{x=**o**;	y=**e**}
{x=**o**;	y=**e**}
{x=**e**;	y=**e**}
{x=**e**;	y=**T**}

Alternative example AI: even/odd integers

- Why did adding **is2** and **mod4** fail to fix the approximation problem in the example?

Alternative example AI: even/odd integers

- Why did adding **is2** and **mod4** fail to fix the approximation problem in the example?
    - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
        - and if X is initially even, then this means that the result is divisible by 4

Alternative example AI: even/odd integers

- Why did adding is2 and mod4 fail to fix the approximation problem in the example?
  - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
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- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**!

Alternative example AI: even/odd integers

- Why did adding is2 and mod4 fail to fix the approximation problem in the example?
  - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
    - and if X is initially even, then this means that the result is divisible by 4
- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**!
- how could we get the right answer on this example?

Alternative example AI: even/odd integers

- Why did adding **is2** and **mod4** fail to fix the approximation problem in the example?
  - the example relies on the fact that for all X, (X + 1) * 2 - 2 = 2X
    - and if X is initially even, then this means that the result is divisible by 4
- **lesson from this example**: most programs rely on complex invariants, and designing an abstract domain that can capture those invariants is **hard**!
- how could we get the right answer on this example?
  - more complex abstract values, e.g., oddTimes2?
  - store the mathematical expression for each variable?

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one more try…

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Yet another lattice element: "**odd2**"

```
      { even, odd } = top
          /          \
   {even}        {odd}
     /      \            |
{mod4} {is2}       |
      \        |        /
      {} = bottom
```

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Yet another lattice element: "**odd2**"

- produced by multiplying an odd number by 2 (i.e., transfer fcn for odd * is2 -> odd2)

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          {even}              {odd}
       /     |     \             |
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- produced by multiplying an odd number by 2 (i.e., transfer fcn for odd * is2 -> odd2)
- **where does it go** in the lattice?
    - ~~a sibling of is2 and mod4?~~
    - between even and is2!

```
          { even, odd } = top
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          {even}            {odd}
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       |        |           /
       \      {is2}        /
        \       |         /
          {} = bottom
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         /    \               |
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     |        |              /
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```

Alternative example AI: another attempt

```
x = 0;
y = read_even();
x = y + 1;
y = 2 * x;
x = y - 2;
y = x / 2;
```

Abstract interpr.

```
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
{x=?;      y=?}
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Abstract interpr.

{x=**e**; y=⊥}
{x=?; y=?}
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Abstract interpr.

{x=**e**;	y=$\perp$}
{x=**e**;	y=**e**}
{x=?;	y=?}
{x=?;	y=?}
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{x=**e**;	y=⊥}
{x=**e**;	y=**e**}
{x=**o**;	y=**e**}
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Abstract interpr.

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$\{x=\mathbf{o};\quad y=\mathbf{e}\}$

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{x=**mod4**;	y=**odd2**}
{x=?;	y=?}

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{x=**e**; y=$\perp$}
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{x=**o**; y=**odd2**}
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{x=**mod4**; y=**e**}

Success!

Formalizing the AI algorithm

The core algorithm for abstract interpretation is the following:

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Using LUB at join points models the fact that the program may **take either branch** of an if statement.

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    a. if the item is a basic block, abstractly execute it using the transfer functions (and abstraction function, if applicable)
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    c. if either a. or b. caused a change, re-add dependent blocks to the worklist

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You may be surprised that it is possible to build an abstract interpretation using (some) infinite-height lattices. Next week, we'll discuss ***widening***, which is the technique for this.

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- that is, each loop will be analyzed at most *k-1* times for each variable in the loop, where *k* is the height of the lattice

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  - LUB is **monotonic**
- that is, each loop will be analyzed at most *k-1* times for each variable in the loop, where *k* is the height of the lattice
- otherwise, loops are just a join point and a back-edge in the CFG

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- **pessimistic** algorithms are also possible
  - start with T everywhere and move downwards in the lattice
  - can be stopped at any time (e.g., when a budget is reached), but answer may not be precise

Another example

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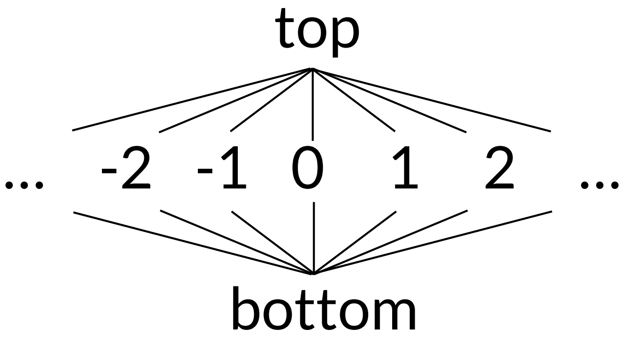

Another example

Consider the following program:

```
w = 5
x = read()
if (x is even)
  y = 5
  w = w + y
else
  y = 10
  w = y
z = y + 1
x = 2 * w
```

(on the whiteboard)

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  - ideally, we'd like $\forall\ x, \gamma(\alpha(x)) = x$
  - but this is too strong: approximation may cause us to lose information! So, the standard formalism is:
    - $\forall\ x, x \in \gamma(\alpha(x))$

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  - ideally, we'd like ∀ x, γ(α(x))
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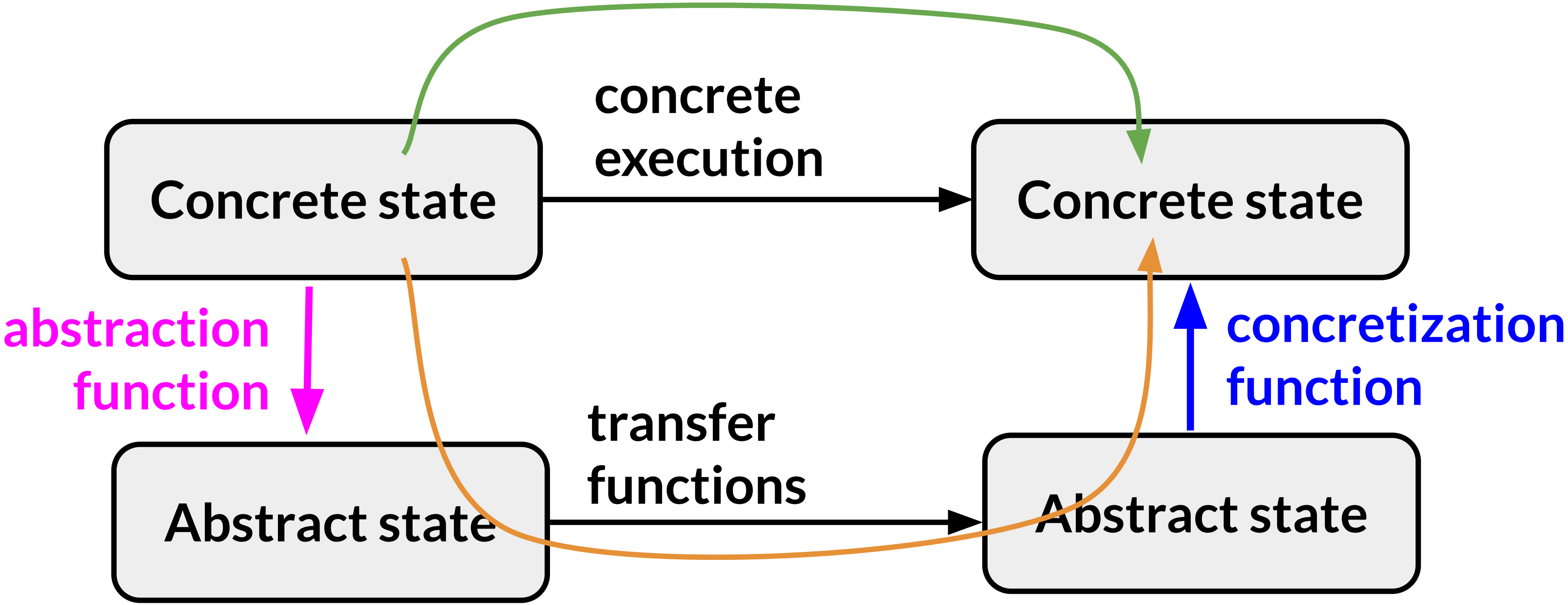
    information! So, the standa
    - ∀ x, x ∈ γ(α(x))

And, it's also necessary to show that the Galois connection holds for the **transfer functions**!

Approximation!

Remember this diagram from earlier?



Do the green and orange paths always lead to the same concrete state?

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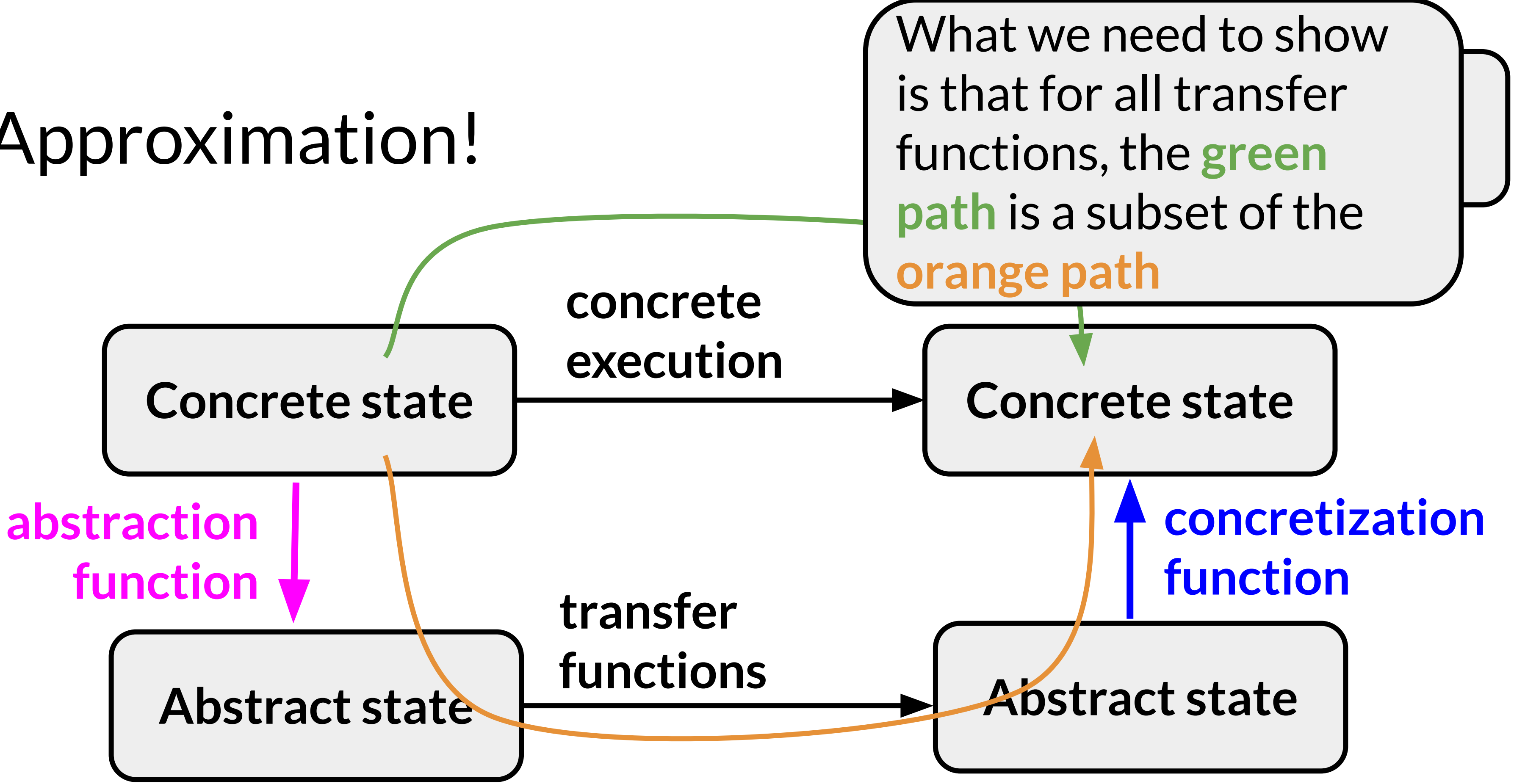

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Course Announcements

- **PA2 (full)** is due next Monday (one week from today!)